



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Multi-kernel unmixing and super-resolution using the Modified Matrix Pencil method

Citation for published version:

Chretien, S & Tyagi, H 2018 'Multi-kernel unmixing and super-resolution using the Modified Matrix Pencil method' ArXiv. <<https://arxiv.org/abs/1807.02862>>

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Publisher's PDF, also known as Version of record

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



Multi-kernel unmixing and super-resolution using the Modified Matrix Pencil method

Stéphane Chréten^{*}
stephane.chretien@npl.co.uk

Hemant Tyagi[†]
htyagi@turing.ac.uk

July 10, 2018

Abstract

Consider L groups of point sources or spike trains, with the l^{th} group represented by $x_l(t)$. For a function $g : \mathbb{R} \rightarrow \mathbb{R}$, let $g_l(t) = g(t/\mu_l)$ denote a point spread function with scale $\mu_l > 0$, and with $\mu_1 < \dots < \mu_L$. With $y(t) = \sum_{l=1}^L (g_l \star x_l)(t)$, our goal is to recover the source parameters given samples of y , or given the Fourier samples of y . This problem is a generalization of the usual super-resolution setup wherein $L = 1$; we call this the multi-kernel unmixing super-resolution problem. Assuming access to Fourier samples of y , we derive an algorithm for this problem for estimating the source parameters of each group, along with precise non-asymptotic guarantees. Our approach involves estimating the group parameters sequentially in the order of increasing scale parameters, i.e., from group 1 to L . In particular, the estimation process at stage $1 \leq l \leq L$ involves (i) carefully sampling the tail of the Fourier transform of y , (ii) a *deflation* step wherein we subtract the contribution of the groups processed thus far from the obtained Fourier samples, and (iii) applying Moitra's modified Matrix Pencil method on a deconvolved version of the samples in (ii).

Key words: Matrix Pencil, super-resolution, unmixing kernels, mixture models, sampling, approximation, signal recovery.

Mathematics Subject Classifications (2010): 15B05, 42A82, 65T99, 65F15, 94A20.

1 Introduction

1.1 Background on super-resolution

Super-resolution consists of estimating a signal x , given blurred observations obtained after convolution with a point spread function g which is assumed to represent the impulse response of the measurement system, such as for eg., a microscope in high density single molecule imaging. Mathematically, x is typically modeled as a superposition of k dirac's, i.e., a *sparse* atomic measure of the form

$$x(t) = \sum_{i=1}^k u_i \delta(t - t_i); \quad u_i \in \mathbb{C}, t_i \in [0, 1),$$

^{*}National Physical Laboratory, Teddington, UK.

[†]Alan Turing Institute, London, UK; School of Mathematics, University of Edinburgh, Edinburgh, UK. This author's work was supported by EPSRC grant EP/N510129/1.

while g is a low pass filter. Denoting

$$y(t) = \sum_{i=1}^k u_i g(t - t_i) \quad (1.1)$$

to be the convolution of x and g , one is usually given information about x either as samples of y , or the Fourier samples of y . This problem has a number of important applications arising for instance in geophysics [26], astronomy [40], medical imaging [21] etc. The reader is referred to [10] for a more comprehensive list of applications. Super-resolution can be seen as a highly non-trivial “off the grid” extension of the finite dimensional sparse estimation problem in compressed sensing [19], [17] and statistics [8]. In the new setting, instead of estimating a sparse vector in a finite dimensional space, the goal is to estimate a sparse measure over the real line \mathbb{R} endowed with its Borel σ -algebra.

Recently, the problem of super-resolution has received a great deal of interest in the signal processing community, triggered by the seminal work of Candès and Fernandez-Granda [10, 9]. They considered the setup where one is given the first few Fourier coefficients of x , i.e., for a cut-off frequency $m \in \mathbb{Z}^+$, we observe $f(s) \in \mathbb{C}$ where

$$f(s) = \int_0^1 e^{i2\pi st} x(dt) = \sum_{i=1}^k u_i e^{i2\pi st_i}; \quad s \in \{-m, -m+1, \dots, m\}. \quad (1.2)$$

Note that this corresponds to taking $g(t) = 2m \text{sinc}(2mt)$ in (1.1), and sampling the Fourier transform of y on the regular grid $\{-m, -m+1, \dots, m\}$. The authors consider the noiseless setting and propose solving an optimization problem over the space of measures which involves minimizing the total variation (TV) norm amongst all measures which are consistent with the observations. The resulting minimization problem is an infinite dimensional convex program over Borel measures on \mathbb{R} . It was shown that the dual of this problem can be formulated as a semi-definite program (SDP) in finitely many variables, and thus can be solved efficiently. Since then, there have been numerous follow-up works such as by Schiebinger et al. [41], Duval and Peyre [14], Denoyelle et al. [13], Bendory et al. [3], Azaïs et al. [2] and many others. For instance, [41] considers the noiseless setting by taking real-valued samples of y with a more general choice of g (such as a Gaussian) and also assumes x to be non-negative. Their proposed approach again involves TV norm minimization with linear constraints. Bendory et al. [3] consider g to be Gaussian or Cauchy, do not place sign assumptions on x , and also analyze TV norm minimization with linear fidelity constraints for estimating x from noiseless samples of y . The approach adopted in [14, 13] is to solve a least-squares-type minimization procedure with a TV norm based penalty term (also referred to as the Beurling LASSO (see for eg., [2])) for recovering x from samples of y . The approach in [15] considers a natural finite approximation on the grid to the continuous problem, and studies the limiting behaviour as the grid becomes finer; see also [16].

From a computational perspective, the aforementioned approaches all admit a finite dimensional dual problem with an infinite number of linear constraints; this is a semi infinite program (SIP) for which there is an extensive literature [23]. For the particular case of non-negative x , Boyd et al. [5] proposed an improved Frank-Wolfe algorithm in the primal. In certain instances, for eg., with Fourier samples (such as in [10, 9]), this SIP can also be reformulated as a SDP. From a practical point of view, SDP is notoriously slow for moderately large number of variables. The algorithm of [5] is a first order scheme with potential local correction steps, and is practically more viable.

From a statistical view point, Candès and Fernandez-Granda [10] showed that their approach exactly recovers x in the noiseless case provided $m \geq 2/\Delta$, where Δ denotes the minimum separation between the spike locations. Similar results for other choices of g were shown by Schiebinger

et al. [41] (for positive measures and without any minimum separation condition), and by Bendory et al. [3] (for signed measures and with a separation condition). In the noisy setting, the state of affairs is radically different since it is known (see for eg., [10, Section 3.2], [35, Corollary 1.4]) that some separation between the spike locations is indeed necessary for stable recovery. When sufficient separation is assumed and provided the noise level is small enough, then stable recovery of x is possible (see for eg., [18, 14, 13, 2]).

When one is given the first few Fourier samples of the spike train x (i.e., (1.2)), then there exist other approaches that can be employed. Prony’s method [12] is a classical method that involves finding the roots of a polynomial, whose coefficients form a null vector of a certain Hankel matrix. Prony’s method and its variants have also been recently studied by Potts and Tasche (for eg., [38, 39]), Plonka and Tasche [37], and others. The matrix pencil (MP) method [24] is another classical approach that involves computing the generalized eigenvalues of a suitable matrix pencil. Both these methods recover x exactly in the absence of noise provided $m \geq k$ (so $2k + 1$ samples), and are also computationally feasible. Recently, Moitra [35, Theorem 2.8] showed that the MP method is stable in the presence of noise provided the noise level is not too large, and $m > \frac{1}{\Delta} + 1$. Moitra also showed [35, Corollary 1.4] that such a dependence is necessary, in the sense that if $m < (1 - \epsilon)/\Delta$, then the noise level would have to be $O(2^{-\epsilon k})$ to be able to stably recover x . More recently, Tang et al. [48, 47] studied approaches based on atomic norm minimization, which can be formulated as a SDP. In [48, Theorem 1.1], the authors considered the signs of the amplitudes of u_j to be generated randomly, with noiseless samples. It was shown that if $m \geq 2/\Delta$, and if the Fourier samples are obtained at $n = \Omega(k \log k \log m)$ indices selected uniformly at random from $\{-m, \dots, m\}$, then x is recovered exactly with high probability. In [47, Theorem 1], the authors considered Gaussian noise in the samples and showed that if $m \geq 2/\Delta$, then the solution returned by the SDP estimates the vector of original Fourier samples (i.e., $(f(-m), \dots, f(m))^T$) at a mean square rate $O(\sigma^2 k \frac{\log m}{m})$, with σ^2 denoting variance of noise. Moreover, they also show [47, Theorem 2] that the spike locations and amplitude terms corresponding to the SDP solution are localized around the true values. Very similar in spirit to the MP method for sum of exponential estimation are the Pony-like methods ESPRIT and MUSIC which can also be used for spike train estimation using the same Fourier domain measurement trick as in [35]. These methods can be interpreted as model reduction methods based on lowrank approximation of Hankel matrices [34]. However, to the best of our knowledge, a precise, finite sample based analysis of ESPRIT, similar in spirit to the analysis of the MP method proposed in [35] is still lacking, although a first step in this direction (using Adamian Arov and Krein theory) can be found in [37].

1.2 Super-resolution with multiple kernels

In this paper, we consider a generalization of the standard super-resolution problem by assuming that the measurement process now involves a *superposition* of convolutions of several spike trains with *different* point spread functions. This problem, which we call the “*multi-kernel unmixing super-resolution problem*” appears naturally in many practical applications such as single molecule spectroscopy [25], spike sorting in neuron activity analysis [28, 6], DNA sequencing [29, 30], spike hunting in galaxy spectra [7, 32] etc.

A problem of particular interest at the National Physical Laboratory, the UK’s national metrology institute, is isotope identification [33, 46] which is of paramount importance in nuclear security. Hand-held radio-isotope identifiers are known to suffer from poor performance [46] and new and precise algorithms have to be devised for this problem. While it is legitimately expected for Radio Isotope Identifier Devices to be accurate across all radioactive isotopes, the US Department of Homeland Security requires all future identification systems to be able to meet a minimum identifi-

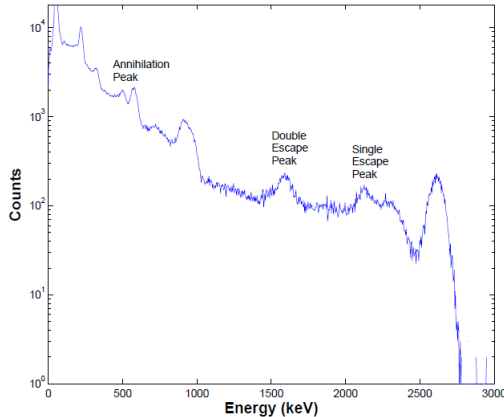


Figure 1: ^{232}Th spectrum showing escape peaks and annihilation peaks [45, p.9]

cation standard set forth in ANSI N42.34. Effective identification from low resolution information is known to be reliably achievable by trained spectroscopists whereas automated identification using standard algorithms sometimes fails up to three fourth of the time [45]. For instance, the spectrum of ^{232}Th is plotted in Figure 1, which is extracted from [45, p.9]. Isotope identification involves the estimation of a set of peak locations in the gamma spectrum where the signal is blurred by convolution with kernels of different window sizes. Mixtures of different spectra can be particularly difficult to analyse using traditional methods and a need for precise unmixing algorithms in such generalised super-resolution problems may play an important role in future applications such as reliable isotope identification.

Another application of interest is DNA sequencing in the vein of [29]. Sequencing is usually performed using some of the variants of the enzymatic method invented by Frederick Sanger [1]. Sequencing is based on enzymatic reactions, electrophoresis, and some detection technique. Electrophoresis is a technique used to separate the DNA sub-fragments produced as the output of four specific chemical reactions, as described in more detail in [29]. DNA fragments are negatively charged in solution. An efficient color-coding strategy has been developed to permit sizing of all four kinds of DNA sub-fragments by electrophoresis in a single lane of a slab gel or in a capillary. In each of the four chemical reactions, the primers are labeled by one of four different fluorescent dyes. The dyes are then excited by a laser based con-focal fluorescent detection system, in a region within the slab gel or capillary. In that process, fluorescence intensities are emitted in four wavelength bands as shown with different color codes in Figure 2 below. However, quoting [4], “*because the electrophoretic process often fails to separate peaks adequately, some form of deconvolution filter must be applied to the data to resolve overlapping events. This process is complicated by the variability of peak shapes, meaning that conventional deconvolution often fails.*” The methods developed in the present paper aim at achieving accurate deconvolution with different peak shapes and might therefore be useful for practical deconvolution problems in DNA sequencing. We will now provide the mathematical formulation of the problem and describe the main idea of our approach along with our contributions, and discuss related work for this problem.

1.3 Problem formulation

Say we have L groups of point sources where $\{t_{l,i}\}_{i=1}^K \subset [0, 1)$ and $(u_{l,i})_{i=1}^K$ (with $u_{l,i} \in \mathbb{C}$) denote the locations and (complex-valued) amplitudes respectively of the sources in the l^{th} group. Our

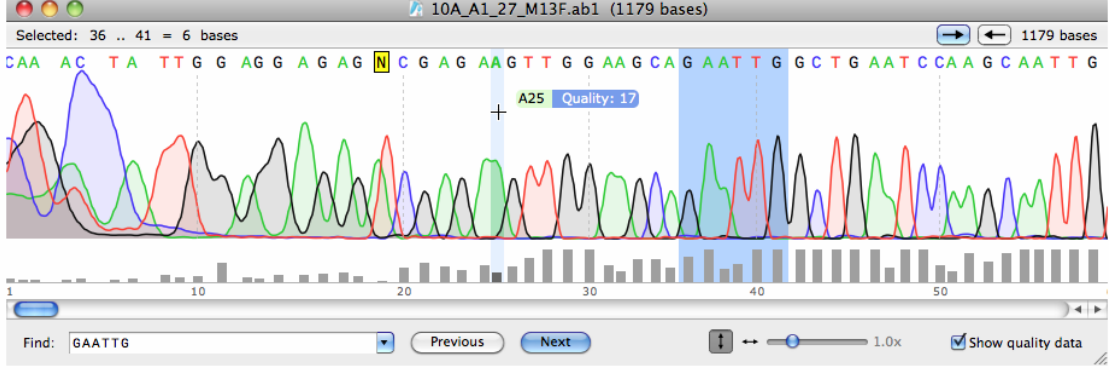


Figure 2: Capture of the result of a sequencing using SnapGene_Viewer [43].

signal of interest is defined as

$$x(t) = \sum_{l=1}^L x_l(t) = \sum_{l=1}^L \left(\sum_{j=1}^K u_{l,j} \delta(t - t_{l,j}) \right).$$

Let $g \in L_1(\mathbb{R})$ be a positive definite function¹ with its Fourier transform $\bar{g}(s) = \int_{\mathbb{R}} g(t) \exp(\iota 2\pi s t) dt$ for $s \in \mathbb{R}$. Consider L distinct kernels $g_l(t) = g(t/\mu_l)$, $l = 1, \dots, L$ where $0 < \mu_1 < \dots < \mu_L$. Let $y(t) = \sum_{l=1}^L (g_l \star x_l)(t) = \sum_{l=1}^L \sum_{j=1}^K u_{l,j} g_l(t - t_{l,j})$ where \star denotes the convolution operator. Let f denote the Fourier transform of y , i.e., $f(s) = \int_{\mathbb{R}} y(t) \exp(\iota 2\pi s t) dt$. Denoting the Fourier transform of g_l by \bar{g}_l , we get

$$f(s) = \sum_{l=1}^L \underbrace{\bar{g}_l(s) \left(\sum_{j=1}^K u_{l,j} \exp(\iota 2\pi s t_{l,j}) \right)}_{f_l(s)}. \quad (1.3)$$

Assuming black box access to the complex valued function f , our aim is to recover estimates of $\{t_{l,i}\}_{i=1}^K$ and $(u_{l,i})_{i=1}^K$ for each $l = 1, \dots, L$ from few *possibly noisy* samples $\tilde{f}(s) = f(s) + w(s)$ of f . Here, $w(s) \in \mathbb{C}$ denotes measurement noise at location s .

Gaussian kernels. For ease of exposition, we will from now on consider g to be a Gaussian, i.e., $g(t) = \exp(-t^2/2)$ so that $g_l(t) = \exp(-t^2/(2\mu_l^2))$, $l = 1, \dots, L$. It is well known that g is positive definite [49, Theorem 6.10], moreover, $\bar{g}_l(s) = \sqrt{2\pi}\mu_l \exp(-2\pi^2 s^2 \mu_l^2)$. We emphasize that our restriction to Gaussian kernels is only to minimize the amount of tedious computations in the proof. However, our proof technique can be extended to handle more general positive definite g .

1.4 Main idea of our work: Fourier tail sampling

To explain the main idea, let us consider the noiseless setting $w(s) = 0$. Our main algorithmic idea stems from observing (1.3), wherein we notice that for s sufficiently large, $f(s)$ is equal to $f_1(s)$ plus a perturbation term arising from the tails of $f_2(s), \dots, f_L(s)$. Thus, $f(s)/\bar{g}_1(s)$ is equal

¹Recall from [49, Theorem 6.11] that a continuous function $g \in L_1(\mathbb{R})$ is positive definite if and only if g is bounded and its Fourier transform is nonnegative and non vanishing.

to $\sum_{j=1}^K u_{l,j} \exp(\iota 2\pi s t_{l,j})$ (which is a weighted sum of complex exponentials) plus a perturbation term. We control this perturbation by choosing s to be sufficiently large, and recover estimates $\hat{t}_{1,j}, \hat{u}_{1,j}$ (up to a permutation ϕ_1) via the Modified Matrix Pencil (MMP) method of Moitra [35] (outlined as Algorithm 1). Given these, we form the estimate $\hat{f}_1(s)$ to $f_1(s)$ where

$$\hat{f}_1(s) = \bar{g}_1(s) \sum_{j=1}^K \hat{u}_{1,\phi_1(j)} \exp(\iota 2\pi s \hat{t}_{1,\phi_1(j)}).$$

Provided the estimates are good enough, we will have $f(s) - \hat{f}_1(s) \approx \sum_{l=2}^L f_l(s)$. Therefore, by applying the above procedure to $f(s) - \hat{f}_1(s)$, we can hope to recover estimates of $t_{2,j}$'s and $u_{2,j}$'s as well. By proceeding recursively, it is clear that we can perform the above procedure to recover estimates to each $\{t_{l,i}\}_{i=1}^K \subset [0, 1)$ and $(u_{l,i})_{i=1}^K$ for all $l = 1, \dots, L$. A delicate issue that needs to be addressed for each intermediate group $1 < l < L$ is the following. While estimating the parameters for group $1 < l < L$, the samples $\frac{f(s) - \sum_{i=1}^{l-1} \hat{f}_i(s)}{\bar{g}_l(s)}$ that we obtain will have perturbation arising due to (a) the tails of $f_{l+1}(s), \dots, f_L(s)$ and, (b) the estimates $\hat{f}_1(s), \dots, \hat{f}_{l-1}(s)$ computed thus far. In particular, going “too deep” in to the tail of $f_l(s)$ would blow up the perturbation term in (b), while not going sufficiently deep would blow up the perturbation term in (a). Therefore, in order to obtain stable estimates to the parameters $t_{l,j}, u_{l,j}$, we will need to control these perturbation terms by carefully choosing the locations, as well as the number of sampling points in the tail.

Further remarks. Our choice of using the MMP method for estimating the spike locations and amplitudes at each stage is primarily dictated by two reasons. Firstly, it is extremely simple to implement in practice. Moreover, it comes with precise quantitative error bounds (see Theorem 3) for estimating the source parameters – especially for the setting of adversarial bounded noise – which fit seamlessly in the theoretical analysis of our method. Of course in practice, other methods tailored to handle sums of complex exponentials (such as ESPRIT, MUSIC etc.) could also be used. To our knowledge, error bounds of the form in Theorem 3 do not currently exist for these other methods.

1.5 Main results

Our algorithm based on the aforementioned idea is outlined as Algorithm 2 which we name KrUMMP (Kernel Unmixing via Modified Matrix Pencil). Our main result for the noiseless setting ($w \equiv 0$) is stated as Theorem 5 in its full generality. We state its following informal version assuming the spike amplitudes in each group to be $\Theta(1)$, and with \lesssim used to hide positive constants. Note that $d_w : [0, 1]^2 \rightarrow [0, 1/2]$ is the usual wrap around distance on $[0, 1]$ (see (2.1))

Theorem 1 (Noiseless case). *Denote $\Delta_l := \min_{i,j} d_w(t_{l,i}, t_{l,j}) > 0$ for each $1 \leq l \leq L$. Let $\varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_L$ satisfy $\varepsilon_l \lesssim \Delta_l/2$ for each l . Moreover, let $\varepsilon_{L-1} \lesssim \alpha \varepsilon_L^2$ and $\varepsilon_l \lesssim \beta_l (\varepsilon_{l+1})^{2(1+\gamma)}$ hold for $1 \leq l \leq L-1$ with $\alpha, \beta_l, \gamma_l > 0$ depending on the problem parameters (see (3.26), (3.28)). Finally, let $m_L = \Theta(1/\Delta_L)$, $s_L = 0$, and*

$$m_l = \Theta(1/\Delta_l), \quad s_l = \Theta \left(m_l + \frac{1}{(\mu_{l+1}^2 - \mu_l^2)^{1/2}} \log^{1/2} \left(\frac{K^{3/2}(L-l)\mu_L}{\varepsilon_l \mu_l} \right) \right); \quad 1 \leq l \leq L-1.$$

Then, for each $l = 1, \dots, L$, there exists a permutation $\phi_l : [K] \rightarrow [K]$ such that

$$d_w(\hat{t}_{l,\phi_l(j)}, t_{l,j}) \leq \varepsilon_l, \quad |\hat{u}_{l,\phi_l(j)} - u_{l,j}| < E_l(\varepsilon_l); \quad j = 1, \dots, K.$$

Here, $E_L(\varepsilon_L) = O(\frac{K}{\Delta_L})$, $E_l(\varepsilon_l) = O\left(\left(\frac{K}{\Delta_l} + \frac{1}{(\mu_{l+1}^2 - \mu_l^2)^{1/2}} \log^{1/2}\left(\frac{K^{3/2}(L-l)\mu_L}{\varepsilon_l\mu_l}\right)\right)\varepsilon_l\right)$ for $1 \leq l \leq L-1$.

The conditions on $\varepsilon \in (0, 1)$ imply that the estimation errors corresponding to group l should be sufficiently smaller than that of group $l+1$. This is because the estimation errors arising in stage l percolate to stage $l+1$, and hence need to be controlled for stable recovery of source parameters for the $(l+1)^{th}$ group. Note that the conditions on s_l (sampling offset at stage l) involve upper and lower bounds for reasons stated in the previous section. If μ_l is close to μ_{l+1} then s_l will have to be suitably large in order to distinguish between \bar{g}_l and \bar{g}_{l+1} , as one would expect intuitively. An interesting feature of the result is that it only depends on separation within a group (specified by Δ_l), and so spikes belonging to different groups are allowed to overlap.

Our second result is for the noisy setting. Say at stage $1 \leq p \leq L$ of Algorithm 2, we observe $\tilde{f}(s) = f(s) + w_p(s)$ where $w_p(s) \in \mathbb{C}$ denotes noise at location s . Our main result for this setting is Theorem 6. Denoting $w_p = (w_p(s_p - m_p), w_p(s_p - m_p + 1), \dots, w_p(s_p + m_p - 1))^T \in \mathbb{C}^{2m_p}$ to be the noise vector at stage p , we state its informal version below assuming the spike amplitudes in each group to be $\Theta(1)$.

Theorem 2 (Noisy case). *Say at stage $1 \leq p \leq L$ of Algorithm 2, we observe $\tilde{f}(s) = f(s) + w_p(s)$ where $w_p(s) \in \mathbb{C}$ denotes noise at location s . For each $1 \leq l \leq L$, let ε_l, m_l, s_l be chosen as specified in Theorem 1. Say $\|w_L\|_\infty \lesssim \mu_L e^{-\mu_L^2/\Delta_L^2} \frac{\varepsilon_L}{\sqrt{K}}$, and also*

$$\|w_l\|_\infty \lesssim \frac{(\varepsilon_l \mu_l)^{1+C(\mu_l, \mu_{l+1}, \Delta_l)}}{\sqrt{K} (K^{3/2} L \mu_L)^{C(\mu_l, \mu_{l+1}, \Delta_l)}}; \quad 2 \leq l \leq L-1$$

where $C(\mu_l, \mu_{l+1}, \Delta_l) > 0$ depends only on $\mu_l, \mu_{l+1}, \Delta_l$. Then, for each $l = 1, \dots, L$, there exists a permutation $\phi_l : [K] \rightarrow [K]$ such that

$$d_w(\hat{t}_{l, \phi_l(j)}, t_{l,j}) \leq \varepsilon_l, \quad |\hat{u}_{l, \phi_l(j)} - u_{l,j}| < E_l(\varepsilon_l); \quad j = 1, \dots, K,$$

where $E_l(\cdot)$ is as in Theorem 1.

2 Notation and Preliminaries

Notation. Vectors and matrices are denoted by lower and upper cases letters respectively. For $n \in \mathbb{N}$, we denote $[n] = \{1, \dots, n\}$. The imaginary unit is denoted by $\iota = \sqrt{-1}$. The ℓ_p ($1 \leq p \leq \infty$) norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|_p$ (defined as $(\sum_i |x_i|^p)^{1/p}$). In particular, $\|x\|_\infty := \max_i |x_i|$. For a matrix $A \in \mathbb{R}^{m \times n}$, we will denote its spectral norm (i.e., largest singular value) by $\|A\|$ and its Frobenius norm by $\|A\|_F$ (defined as $(\sum_{i,j} A_{i,j}^2)^{1/2}$). For positive numbers a, b , we denote $a \lesssim b$ to mean that there exists an absolute constant $C > 0$ such that $a \leq Cb$. The wrap around distance on $[0, 1]$ is denoted by $d_w : [0, 1]^2 \rightarrow [0, 1/2]$ where we recall that

$$d_w(t_1, t_2) = \min\{|t_1 - t_2|, 1 - |t_1 - t_2|\}. \quad (2.1)$$

2.1 Matrix Pencil (MP) method

We now review the classical Matrix Pencil (MP) method for estimating positions of point sources from Fourier samples. Consider the signal $x(t) := \sum_{j=1}^K u_j \delta(t - t_j)$ where $u_j \in \mathbb{C}$, $t_j \in [0, 1)$ are

unknown. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be the Fourier transform of x so that $f(s) = \sum_{j=1}^K u_j \exp(\iota 2\pi s t_j)$. For any given offset $s_0 \in \mathbb{Z}^+$, let $s = s_0 + i$ where $i \in \mathbb{Z}$; clearly

$$f(s_0 + i) = \sum_{j=1}^K u_j \exp(\iota 2\pi(s_0 + i)t_j) = \sum_{j=1}^K u'_j \exp(\iota 2\pi i t_j)$$

where $u'_j = u_j \exp(\iota 2\pi s_0 t_j)$, $j = 1, \dots, K$. Choose $i \in \{-m, -m+1, \dots, m-1\}$ to form the $m \times m$ matrices

$$H_0 = \begin{bmatrix} f(s_0) & f(s_0 + 1) & \cdots & f(s_0 + m - 1) \\ f(s_0 - 1) & f(s_0) & \cdots & f(s_0 + m - 2) \\ \vdots & \vdots & & \vdots \\ f(s_0 - m + 1) & f(s_0 - m + 2) & \cdots & f(s_0) \end{bmatrix} \quad (2.2)$$

and

$$H_1 = \begin{bmatrix} f(s_0 - 1) & f(s_0) & \cdots & f(s_0 + m - 2) \\ f(s_0 - 2) & f(s_0 - 1) & \cdots & f(s_0 + m - 3) \\ \vdots & \vdots & & \vdots \\ f(s_0 - m) & f(s_0 - m + 1) & \cdots & f(s_0 - 1) \end{bmatrix}. \quad (2.3)$$

Denoting $\alpha_j = \exp(-\iota 2\pi t_j)$ for $j = 1, \dots, K$, and the Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_K \\ \vdots & \vdots & & \vdots \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_K^{m-1} \end{bmatrix}, \quad (2.4)$$

clearly $H_0 = V D_{u'} V^H$ and $H_1 = V D_{u'} D_\alpha V^H$. Here, $D_{u'} = \text{diag}(u'_1, \dots, u'_K)$ and $D_\alpha = \text{diag}(\alpha_1, \dots, \alpha_K)$ are diagonal matrices. One can readily verify² that the K non zero generalized eigenvalues of (H_1, H_0) are equal to the α_j 's. Hence by forming the matrices H_0, H_1 , we can recover the unknown t_j 's exactly from $2K$ samples of f . Once the t_j 's are recovered, we can recover the u'_j 's exactly, as the solution of the linear system

$$\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(m-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_K \\ \vdots & \vdots & & \vdots \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_K^{m-1} \end{bmatrix} \begin{bmatrix} u'_1 \\ \vdots \\ u'_K \end{bmatrix}.$$

Thereafter, the u_j 's are found as $u_j = u'_j / \exp(\iota 2\pi s_0 t_j)$.

2.2 The Modified Matrix Pencil (MMP) method

We now consider the noisy version of the setup defined in the previous section. For a collection of K point sources with parameters $u_j \in \mathbb{C}$, $t_j \in [0, 1]$, we are given noisy samples

$$\tilde{f}(s) = f(s) + \eta_s, \quad (2.5)$$

²Indeed, $H_0(V^H)^\dagger = V D_{u'}$ and so $H_1(V^H)^\dagger = V D_{u'} D_\alpha = H_0(V^H)^\dagger D_\alpha$.

for $s \in \mathbb{Z}$ and where $\eta_s \in \mathbb{C}$ denotes noise.

Let us choose $s = s_0 + i$ for a given offset $s_0 \in \mathbb{Z}^+$, and $i = \{-m, -m+1, \dots, m-1\}$ for a positive integer m . Using $(\tilde{f}(s_0 + i))_{i=-m}^{m-1}$, let us form the matrices $\tilde{H}_0, \tilde{H}_1 \in \mathbb{C}^{m \times m}$ as in (2.2), (2.3). We now have $\tilde{H}_0 = H_0 + E$, $\tilde{H}_1 = H_1 + F$ where H_0, H_1 are as defined in (2.2), (2.3), and

$$E = \begin{bmatrix} \eta_0 & \eta_1 & \dots & \eta_{m-1} \\ \eta_{-1} & \eta_0 & \dots & \eta_{m-2} \\ \vdots & \vdots & & \vdots \\ \eta_{-(m-1)} & \eta_{-(m-2)} & \dots & \eta_0 \end{bmatrix}, \quad F = \begin{bmatrix} \eta_{-1} & \eta_0 & \dots & \eta_{m-2} \\ \eta_{-2} & \eta_{-1} & \dots & \eta_{m-3} \\ \vdots & \vdots & & \vdots \\ \eta_{-m} & \eta_{-(m-1)} & \dots & \eta_{-1} \end{bmatrix}.$$

represent the perturbation matrices. Algorithm 1 namely the Modified Matrix Pencil (MMP) method [35] outlines how we can recover \hat{t}_j, \hat{u}_j for $j = 1, \dots, K$.

Before proceeding we need to make some definitions. Let $u_{\max} = \max_j |u_j|$ and $u_{\min} = \min_j |u_j|$. We denote the largest and smallest non zero singular values of V by $\sigma_{\max}, \sigma_{\min}$ respectively, and the condition number of V by κ where $\kappa = \sigma_{\max}/\sigma_{\min}$. Let $\eta_{\max} := \max_i |\eta_i|$. We will define Δ as the minimum separation between the locations of the point sources where $\Delta := \min_{j \neq j'} d_w(t_j, t_{j'})$.

Algorithm 1 Modified Matrix Pencil (MMP) method [35]

- 1: **Input:** $K, m, s_0, \tilde{H}_0, \tilde{H}_1$.
 - 2: **Output:** $\hat{u}_j, \hat{t}_j; j = 1, \dots, K$.
 - 3: Let $\hat{U} \in \mathbb{C}^{m \times K}$ be the top K singular vector matrix of \tilde{H}_0 .
 - 4: Let $\hat{A} = \hat{U}^H \tilde{H}_0 \hat{U}$ and $\hat{B} = \hat{U}^H \tilde{H}_1 \hat{U}$.
 - 5: Find generalized eigenvalues $(\hat{\lambda}_j)_{j=1}^n$ of (\hat{B}, \hat{A}) .
 - 6: Let $\hat{\alpha}_j = \exp(-i2\pi\hat{t}_j) = \hat{\lambda}_j/|\hat{\lambda}_j|$ where $\hat{t}_j \in [0, 1)$.
 - 7: Form Vandermonde matrix $\hat{V} \in \mathbb{C}^{m \times K}$ using $\hat{\alpha}_j$'s.
 - 8: Find $\hat{u}' = \hat{V}^\dagger v$ where $v = [f(s_0) \ f(s_0 + 1) \ \dots \ f(s_0 + m - 1)]^T$.
 - 9: Find $\hat{u}_j = \hat{u}'_j \exp(-i2\pi s_0 \hat{t}_j); j = 1, \dots, K$.
-

The following theorem is a more precise version of [35, Theorem 2.8], with the constants computed explicitly. Moreover, the result in [35, Theorem 2.8] was specifically for the case $s_0 = 0$. We outline Moitra's proof in Appendix A.1 for completeness and fill in additional details (using auxiliary results from Appendix B) to arrive at the statement of Theorem 3.

Theorem 3 ([35]). *For $0 \leq \varepsilon < \Delta/2$, say $m > \frac{1}{\Delta-2\varepsilon} + 1$. Moreover, for $C = 10 + \frac{1}{2\sqrt{2}}$, say*

$$\eta_{\max} \leq \varepsilon \frac{u_{\min} \sigma_{\min}^2}{2mC\sqrt{K}} \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}} \right)^{-1}. \quad (2.6)$$

Then, there exists a permutation $\phi : [K] \mapsto [K]$ such that the output of the MMP method satisfies for each $i = 1, \dots, K$

$$d_w(\hat{t}_{\phi(i)}, t_i) \leq \varepsilon, \quad \|\hat{u}_\phi - u\|_\infty \leq \left(\frac{2\pi m^{3/2} K u_{\max} + \frac{u_{\min} \sigma_{\min}^2}{2C\sqrt{mK}} \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}} \right)^{-1}}{(m - \frac{1}{\Delta-2\varepsilon} - 1)^{1/2}} + 2\pi u_{\max} s_0 \right) \varepsilon \quad (2.7)$$

where \hat{u}_ϕ is formed by permuting the indices of \hat{u} w.r.t ϕ .

The following Corollary of Theorem 3 simplifies the expression for the bound on $\|\hat{u}_\phi - u\|_\infty$ in (2.7), and will be useful for our main results later on. The proof is deferred to Appendix A.2.

Corollary 1. *For $0 \leq \varepsilon < c\Delta/2$ where $c \in [0, 1)$ is a constant, say $\frac{2}{\Delta-2\varepsilon} + 1 < m \leq \frac{2}{(1-c)\Delta} + 1$. Moreover, for $C = 10 + \frac{1}{2\sqrt{2}}$, say*

$$\eta_{\max} \leq \varepsilon \frac{u_{\min}}{5C\sqrt{K}} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right)^{-1}. \quad (2.8)$$

Then, there exists a permutation $\phi : [K] \mapsto [K]$ such that the output of the MMP method satisfies for each $i = 1, \dots, K$

$$d_w(\hat{t}_{\phi(i)}, t_i) \leq \varepsilon, \quad \|\hat{u}_\phi - u\|_\infty < (\tilde{C} + 2\pi u_{\max} s_0) \varepsilon \quad (2.9)$$

where $\tilde{C} = 4\pi K u_{\max} \left(\frac{2}{\Delta(1-c)} + 1\right) + \frac{u_{\min}}{C\sqrt{K}} \left(1 + 16 \frac{u_{\max}^2}{u_{\min}^2}\right)^{-1}$, and \hat{u}_ϕ is formed by permuting the indices of \hat{u} w.r.t ϕ .

3 Unmixing Gaussians in Fourier domain: Noiseless case

We now turn our attention to the main focus of this paper, namely that of unmixing Gaussians in the Fourier domain. We will in general assume the Fourier samples to be noisy as in (2.5), with f defined in (1.3). In this section, we will focus on the noiseless setting wherein $\tilde{f}(s) = f(s)$ for each s . The noisy setting is analyzed in the next section.

Let us begin by noting that when $L = 1$, i.e., in the case of a single kernel, the problem is solved easily. Indeed, we have from (1.3) that $f(s) = \bar{g}_1(s) \sum_{j=1}^K u_{1,j} \exp(\iota 2\pi s t_{1,j})$. Clearly, one can exactly recover $(t_{1,j})_{j=1}^K \in [0, 1)$ and $(u_{1,j})_{j=1}^K \in \mathbb{C}$ via the MP method by first obtaining the samples $f(-m)$, $f(-m+1)$, \dots , $f(m-1)$, and then working with $f(s)/\bar{g}_1(s)$.

The situation for the case $L \geq 2$ is however more delicate. Before proceeding, we need to make some definitions and assumptions.

- We will denote $u_{\max} = \max_{l,j} |u_{l,j}|$ and $u_{\min} = \min_{l,j} |u_{l,j}|$.
- The sources in the l^{th} group are assumed to have a minimum separation of

$$\Delta_l := \min_{i \neq j} d_w(t_{l,i}, t_{l,j}) > 0.$$

- Denoting $\alpha_{l,j} = \exp(-i2\pi t_{l,j})$, $V_l \in \mathbb{C}^{m_l \times K}$ will denote the Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \alpha_{l,1} & \alpha_{l,2} & \dots & \alpha_{l,K} \\ \vdots & \vdots & & \vdots \\ \alpha_{l,1}^{m_l-1} & \alpha_{l,2}^{m_l-1} & \dots & \alpha_{l,K}^{m_l-1} \end{bmatrix}$$

for each $l = 1, \dots, L$ analogous to (2.4). $\sigma_{\max,l}, \sigma_{\min,l}$ will denote its largest and smallest non-zero singular values, and $\kappa_l = \sigma_{\max,l}/\sigma_{\min,l}$ its condition number. Recall from Theorem 8 that if $m_l > \frac{1}{\Delta_l} + 1$, then $\kappa_l^2 \leq \frac{m_l + \frac{1}{\Delta_l} - 1}{m_l - \frac{1}{\Delta_l} - 1}$. In particular, $\sigma_{\max,l}^2 \leq m_l + \frac{1}{\Delta_l} - 1$ and $\sigma_{\min,l}^2 \geq m_l - \frac{1}{\Delta_l} - 1$.

Algorithm 2 Kernel Unmixing via Modified Matrix Pencil (KrUMMP)

- 1: **Input:** $K, m_l, s_l, \mu_l; l = 1, \dots, L$.
 - 2: **Initialize:** $\hat{u}_{l,j}, \hat{t}_{l,j} = 0; l = 1, \dots, L; j = 1, \dots, K$. Also, $\hat{f}_1 \equiv 0$.
 - 3: **Output:** $\hat{u}_{l,j}, \hat{t}_{l,j}; l = 1, \dots, L; j = 1, \dots, K$.
 - 4: **for** $l = 1, \dots, L$ **do**
 - 5: Obtain samples $\frac{\tilde{f}(s_l+i) - \sum_{j=1}^{l-1} \hat{f}_j(s_l+i)}{\bar{g}_l(s_l+i)}$ for $i = -m_l, \dots, m_l - 1$.
 - 6: Form $\tilde{H}_0^{(l)}, \tilde{H}_1^{(l)} \in \mathbb{C}^{m_l \times m_l}$ using the above samples as in (2.2), (2.3).
 - 7: Input $\tilde{H}_0^{(l)}, \tilde{H}_1^{(l)}$ to MMP algorithm and obtain estimates $(\hat{u}_{l,j})_{j=1}^K, (\hat{t}_{l,j})_{j=1}^K$.
 - 8: Define $\hat{f}_l: \mathbb{R} \rightarrow \mathbb{C}$ as $\hat{f}_l(s) := \bar{g}_l(s) \sum_{j=1}^K \hat{u}_{l,j} \exp(\iota 2\pi s \hat{t}_{l,j})$.
 - 9: **end for**
-

3.1 The case of two kernels

We first consider the case of two Gaussian kernels as the analysis here is relatively easier to digest compared to the general case. Note that f is now of the form

$$f(s) = \bar{g}_1(s) \left(\sum_{j=1}^K u_{1,j} \exp(\iota 2\pi s t_{1,j}) \right) + \bar{g}_2(s) \left(\sum_{j=1}^K u_{2,j} \exp(\iota 2\pi s t_{2,j}) \right)$$

where we recall $\bar{g}_l(s) = \sqrt{2\pi} \mu_l \exp(-2\pi^2 s^2 \mu_l^2); l = 1, 2$. The following theorem provides sufficient conditions on the choice of the sampling parameters for approximate recovery of $t_{1,j}, t_{2,j}, u_{1,j}, u_{2,j}$ for each $j = 1, \dots, K$.

Theorem 4. Let $0 \leq \varepsilon_2 < c\Delta_2/2$ for a constant $c \in [0, 1)$, $s_2 = 0$, and $m_2 \in \mathbb{Z}^+$ satisfy $\frac{2}{\Delta_2 - 2\varepsilon_2} + 1 \leq m_2 < \frac{2}{\Delta_2(1-c)} + 1$ ($= M_{2,up}$). For constant C as in Theorem 3, let $\varepsilon_1 < c\Delta_1/2$ also satisfy

$$\begin{aligned} & e^{(2\pi^2 M_{2,up}^2 (\mu_2^2 - \mu_1^2))} \left(2\pi u_{max} M_{2,up} + \bar{C}_1 + \bar{C}_2 \log^{1/2} \left(\frac{\bar{C}_3}{\varepsilon_1} \right) \right) \varepsilon_1 \\ & \leq \varepsilon_2 \frac{u_{min} \mu_2}{5C \mu_1 K^{3/2}} \left(1 + 48 \frac{u_{max}}{u_{min}} \right)^{-1}, \end{aligned} \quad (3.1)$$

where $\bar{C}_1, \bar{C}_2, \bar{C}_3 > 0$ are constants depending (see (3.12), (3.13)) only on c, C, \tilde{C}_1 (see (3.2)), $u_{max}, u_{min}, \mu_1, \mu_2, \Delta_1, K$ and a constant $\tilde{c} > 1$. Say $m_1, s_1 \in \mathbb{Z}^+$ are chosen to satisfy

$$\frac{2}{\Delta_1 - 2\varepsilon_1} + 1 \leq m_1 < \frac{2}{\Delta_1(1-c)} + 1 \quad (= M_{1,up}), \quad S_1 \leq s_1 \leq \tilde{c}S_1,$$

where $S_1 = m_1 + \frac{1}{(2\pi^2(\mu_2^2 - \mu_1^2))^{1/2}} \log^{1/2} \left(\frac{5CK^{3/2}u_{max}\mu_2}{u_{min}\mu_1\varepsilon_1} \left(1 + 48 \frac{u_{max}}{u_{min}} \right) \right)$. Then, there exist permutations $\phi_1, \phi_2: [K] \rightarrow [K]$ such that for $j = 1, \dots, K$,

$$\begin{aligned} d_w(\hat{t}_{1,\phi_1(j)}, t_{1,j}) &\leq \varepsilon_1, \quad |\hat{u}_{1,\phi_1(j)} - u_{1,j}| < \left(\bar{C}_1 + \bar{C}_2 \log^{1/2} \left(\frac{\bar{C}_3}{\varepsilon_1} \right) \right) \varepsilon_1, \\ d_w(\hat{t}_{2,\phi_2(j)}, t_{2,j}) &\leq \varepsilon_2, \quad |\hat{u}_{2,\phi_2(j)} - u_{2,j}| < \tilde{C}_2 \varepsilon_2, \end{aligned}$$

where

$$\tilde{C}_l = 4\pi K u_{max} M_{l,up} + \frac{u_{min}}{C\sqrt{K}} \left(1 + 16 \frac{u_{max}^2}{u_{min}^2} \right)^{-1}; \quad l = 1, 2. \quad (3.2)$$

Interpreting Theorem 4. Before proceeding to the proof, we make some useful observations.

- (a) We first choose the sampling parameters ε (accuracy), m (number of samples), s (offset) for the inner kernel \bar{g}_2 and then the outer kernel, i.e., \bar{g}_1 . For group i ($= 1, 2$), we first choose ε_i , then m_i (depending on ε_i), and finally the offset s_i (depending on m_i, ε_i).
- (b) The choice of ε_2 is free, but the choice of ε_1 is constrained by ε_2 as seen from (3.1). In particular, ε_1 needs to be sufficiently small with respect to ε_2 so that the perturbation arising due to the estimation errors for group 1 are controlled when we estimate the parameters for group 2.
- (c) The lower bound on s_1 ensures that we are sufficiently deep in the tail of \bar{g}_2 , so that its effect is negligible. The upper bound on s_1 is to control the estimation errors of the source amplitudes for group 1 (see (2.9)). Observe that $s_2 = 0$ since \bar{g}_2 is the innermost kernel, and so there is no perturbation arising due to the tail of any other inner kernel.
- (d) In theory, ε_1 can be chosen to be arbitrarily close to zero; however, this would result in the offset s_1 becoming large. Consequently, this will lead to numerical errors when we divide by $\bar{g}_1(s_1 + i)$; $i = -m_1, \dots, m_1 - 1$ while estimating the source parameters for group 1.

Order wise dependencies. The Theorem is heavy in notation, so it would help to understand the order wise dependencies of the terms involved. Assume $u_{\max}, u_{\min} = \Theta(1)$. We have $\bar{C}_1 = \Theta(K/\Delta_1)$ and $\bar{C}_2 = \Theta(K/\Delta_2)$ which leads to

$$\begin{aligned}\bar{C}_1 &= \tilde{C}_1 + \Theta\left(\frac{1}{\Delta_1}\right) = \Theta\left(\frac{K}{\Delta_1}\right), \\ \bar{C}_2 &= \Theta\left(\frac{1}{(\mu_2^2 - \mu_1^2)^{1/2}}\right), \quad \bar{C}_3 = \Theta\left(\frac{K^{3/2}\mu_2}{\mu_1}\right).\end{aligned}$$

- (a) For group 2, we have $\varepsilon_2 = O(\Delta_2)$, $m_2 = \Theta(1/\Delta_2)$, $s_2 = 0$, and

$$d_w(\hat{t}_{2,\phi_2(j)}, t_{2,j}) \leq \varepsilon_2, \quad |\hat{u}_{2,\phi_2(j)} - u_{2,j}| = O\left(\frac{K}{\Delta_2}\varepsilon_2\right).$$

- (b) For group 1, $\varepsilon_1 = O(\Delta_1)$ and (3.1) translates to

$$\left(\frac{1}{\Delta_2} + \frac{K}{\Delta_1} + \frac{1}{(\mu_2^2 - \mu_1^2)^{1/2}} \log^{1/2}\left(\frac{K^{3/2}\mu_2}{\mu_1\varepsilon_1}\right)\right) \varepsilon_1 \lesssim \varepsilon_2 \left(\frac{\mu_2}{\mu_1 K^{3/2}}\right) \exp\left(-\frac{2\pi^2(\mu_2^2 - \mu_1^2)}{\Delta_2^2}\right). \quad (3.3)$$

Moreover, $m_1 = \Theta(1/\Delta_1)$ and $s_1 = \Theta(\frac{1}{\Delta_1} + (\mu_2^2 - \mu_1^2)^{-1/2} \log^{1/2}(\frac{K^{3/2}\mu_2}{\mu_1\varepsilon_1}))$. Finally,

$$d_w(\hat{t}_{1,\phi_1(j)}, t_{1,j}) \leq \varepsilon_1, \quad |\hat{u}_{1,\phi_1(j)} - u_{1,j}| = O\left(\left(\frac{K}{\Delta_1} + \frac{1}{(\mu_2^2 - \mu_1^2)^{1/2}} \log^{1/2}\left(\frac{K^{3/2}\mu_2}{\mu_1\varepsilon_1}\right)\right) \varepsilon_1\right). \quad (3.4)$$

Condition on $\varepsilon_1, \varepsilon_2$. It is not difficult to verify that a sufficient condition for (3.3) to hold is that for any given $\theta \in (0, 1/2)$, it holds that

$$\varepsilon_1 \lesssim \varepsilon_2^{\frac{1}{1-\theta}} C(\mu_1, \mu_2, \Delta_1, \Delta_2, K, \theta), \quad (3.5)$$

where $C(\mu_1, \mu_2, \Delta_1, \Delta_2, K, \theta) > 0$ depends only on the indicated parameters. This is outlined in Appendix C for Theorem 5 for the case of L kernels. In other words, ε_1 would have to be sufficiently small with respect to ε_2 , so that the estimation errors carrying forward from the first group to the estimation of the parameters for the second group, are controlled.

Effect of separation between μ_1, μ_2 . Note that as $\mu_1 \rightarrow \mu_2$, then (3.3) becomes more and more difficult to satisfy; in particular, $C(\mu_1, \mu_2, \Delta_1, \Delta_2, K, \theta) \rightarrow 0$ in (3.5). Hence, we would have to sample sufficiently deep in the tail of \bar{g}_1 in order to distinguish \bar{g}_1, \bar{g}_2 as one would intuitively expect. Next, for fixed μ_2 as $\mu_1 \rightarrow 0$, we see that (3.3) becomes easier to satisfy. This is because $\bar{g}_1(s)$ is now small for all s , and hence the perturbation error arising from stage 1 reduces accordingly. However, notice that s_1 now has to increase correspondingly in order to distinguish between \bar{g}_1, \bar{g}_2 (since $\bar{g}_1(s) \approx 0$ for all s). Therefore, in order to control the estimation error of the amplitudes (see (3.4)), ε_1 now has to reduce accordingly. For instance, $\varepsilon_1 = o(\mu_1^{1/3})$ suffices. On the other hand, for fixed μ_1 , as $\mu_2 \rightarrow \infty$, satisfying (3.3) becomes more and more difficult. This is because the tail of \bar{g}_2 becomes thinner, and so, the deconvolution step at stage 2 blows up the error arising from stage 1.

Proof of Theorem 4. The proof is divided into two steps below.

- **Recovering source parameters for first group.** For offset parameter $s_1 \in \mathbb{Z}^+$ (the choice of which will be made clear later), we obtain the samples $(f(s_1 + i))_{i=-m_1}^{m_1-1}$. Now, for any $i = -m_1, \dots, m_1 - 1$, we have that

$$\begin{aligned} \frac{f(s_1 + i)}{\bar{g}_1(s_1 + i)} &= \sum_{j=1}^K u_{1,j} \exp(\iota 2\pi(s_1 + i)t_{1,j}) + \frac{\bar{g}_2(s_1 + i)}{\bar{g}_1(s_1 + i)} \sum_{j=1}^K u_{2,j} \exp(\iota 2\pi(s_1 + i)t_{2,j}) \\ &= \sum_{j=1}^K \underbrace{u_{1,j} \exp(\iota 2\pi s_1 t_{1,j})}_{u'_{1,j}} \exp(\iota 2\pi i t_{1,j}) \\ &\quad + \underbrace{\frac{\mu_2}{\mu_1} \exp(-2\pi^2(s_1 + i)^2(\mu_2^2 - \mu_1^2)) \sum_{j=1}^K u_{2,j} \exp(\iota 2\pi(s_1 + i)t_{2,j})}_{\eta'_{1,i}} \\ &= \sum_{j=1}^K u'_{1,j} \exp(\iota 2\pi i t_{1,j}) + \eta'_{1,i}. \end{aligned} \quad (3.6)$$

Here, $\eta'_{1,i} \in \mathbb{C}$ corresponds to “perturbation” arising from the tail of $f_2(s)$. Since the stated choice of s_1 implies $s_1 > m_1$, this means $\min_i (s_1 + i)^2 = (s_1 - m_1)^2$, and hence clearly

$$|\eta'_{1,i}| \leq \frac{\mu_2}{\mu_1} \exp(-2\pi^2(s_1 - m_1)^2(\mu_2^2 - \mu_1^2)) K u_{\max}, \quad i = -m_1, \dots, m_1 - 1.$$

From (3.6), we can see that $\tilde{H}_0^{(1)} = V_1 D_{u'_1} V_1^H + E^{(1)}$ and $\tilde{H}_1^{(1)} = V_1 D_{u'_1} D_{\alpha_1} V_1^H + F^{(1)}$. Here, $D_{u'_1} = \text{diag}(u'_{1,1}, \dots, u'_{1,K})$ and $D_{\alpha_1} = \text{diag}(\alpha_{1,1}, \dots, \alpha_{1,K})$, while $E^{(1)}, F^{(1)}$ denote the perturbation matrices consisting of $(\eta'_{1,i})_{i=-m_1}^{m_1-1}$ terms, as in (2.2), (2.3).

We obtain estimates $\hat{t}_{1,j}, \hat{u}_{1,j}, j = 1, \dots, K$ via the MMP method. Invoking Corollary 1, we have for $\varepsilon_1 < c\Delta_1/2$, and $\frac{2}{\Delta_1 - 2\varepsilon_1} + 1 \leq m_1 < \frac{2}{\Delta_1(1-c)} + 1 (= M_{1,\text{up}})$ that if s_1 satisfies

$$\frac{\mu_2}{\mu_1} \exp(-2\pi^2(s_1 - m_1)^2(\mu_2^2 - \mu_1^2)) K u_{\max} \leq \varepsilon_1 \frac{u_{\min}}{5C\sqrt{K}} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right)^{-1}, \quad (3.7)$$

$$(3.8)$$

then there exists a permutation $\phi_1 : [K] \rightarrow [K]$ such that for each $j = 1, \dots, K$,

$$d_w(\hat{t}_{1,\phi_1(j)}, t_{1,j}) \leq \varepsilon_1, \quad |\hat{u}_{1,\phi_1(j)} - u_{1,j}| < (\tilde{C}_1 + 2\pi u_{\max} s_1) \varepsilon_1, \quad (3.9)$$

where $\tilde{C}_1 = 4\pi K u_{\max} M_{1,\text{up}} + \frac{u_{\min}}{C\sqrt{K}} \left(1 + 16 \frac{u_{\max}^2}{u_{\min}^2}\right)^{-1}$. Clearly, the condition³

$$s_1 \geq m_1 + \frac{1}{(2\pi^2(\mu_2^2 - \mu_1^2))^{1/2}} \log^{1/2} \left(\frac{5CK^{3/2} u_{\max} \mu_2}{u_{\min} \mu_1 \varepsilon_1} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right) \right)$$

implies (3.7). Moreover, since $s_1 \leq \tilde{c}S_1$ and $m_1 < M_{1,\text{up}}$, we obtain

$$s_1 < \tilde{c} \left(M_{1,\text{up}} + \frac{1}{(2\pi^2(\mu_2^2 - \mu_1^2))^{1/2}} \log^{1/2} \left(\frac{5CK^{3/2} u_{\max} \mu_2}{u_{\min} \mu_1 \varepsilon_1} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right) \right) \right). \quad (3.10)$$

Plugging (3.10) into (3.9) leads to the bound

$$|\hat{u}_{1,\phi_1(j)} - u_{1,j}| < \left(\bar{C}_1 + \bar{C}_2 \log^{1/2} \left(\frac{\bar{C}_3}{\varepsilon_1} \right) \right) \varepsilon_1; \quad j = 1, \dots, K, \quad (3.11)$$

where $\bar{C}_1, \bar{C}_2, \bar{C}_3 > 0$ are constants defined as follows.

$$\bar{C}_1 = \tilde{C}_1 + 2\pi u_{\max} \tilde{c} M_{1,\text{up}}, \quad (3.12)$$

$$\bar{C}_2 = \frac{2\pi u_{\max} \tilde{c}}{(2\pi^2(\mu_2^2 - \mu_1^2))^{1/2}}, \quad \bar{C}_3 = \frac{5CK^{3/2} u_{\max} \mu_2}{u_{\min} \mu_1} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right). \quad (3.13)$$

- **Recovering source parameters for second group.** Let \hat{f}_1 denote the estimate of f_1 obtained using the estimates $(\hat{u}_{1,j})_{j=1}^K, (\hat{t}_{1,j})_{j=1}^K$, defined as

$$\hat{f}_1(s) = \bar{g}_1(s) \sum_{j=1}^K \hat{u}_{1,j} \exp(\iota 2\pi \hat{t}_{1,j} s).$$

For suitable $s_2, m_2 \in \mathbb{Z}_+$ (choice to be made clear later), we now obtain samples

$$\frac{f(s_2 + i) - \hat{f}_1(s_2 + i)}{\bar{g}_2(s_2 + i)}; \quad i = -m_2, \dots, m_2 - 1.$$

³Here, we use the notation $\log^{1/2}(\cdot)$ to denote $|\log(|\cdot|)|^{1/2}$.

Let us note that

$$\begin{aligned}
\frac{f(s_2 + i) - \widehat{f}_1(s_2 + i)}{\bar{g}_2(s_2 + i)} &= \frac{\bar{g}_1(s_2 + i)}{\bar{g}_2(s_2 + i)} \sum_{j=1}^K (u_{1,j} \exp(\iota 2\pi(s_2 + i)t_{1,j}) - \widehat{u}_{1,\phi_1(j)} \exp(\iota 2\pi(s_2 + i)\widehat{t}_{1,\phi_1(j)})) \\
&\quad + \sum_{j=1}^K u_{2,j} \exp(\iota 2\pi(s_2 + i)t_{2,j}) \\
&= \underbrace{\frac{\mu_1}{\mu_2} \exp(2\pi^2(s_2 + i)^2(\mu_2^2 - \mu_1^2)) \sum_{j=1}^K (u_{1,j} \exp(\iota 2\pi(s_2 + i)t_{1,j}) - \widehat{u}_{1,\phi_1(j)} \exp(\iota 2\pi(s_2 + i)\widehat{t}_{1,\phi_1(j)}))}_{\eta'_{2,i}} \\
&\quad + \sum_{j=1}^K \underbrace{u_{2,j} \exp(\iota 2\pi s_2 t_{2,j})}_{u'_{2,j}} \exp(\iota 2\pi i t_{2,j}) \\
&= \sum_{j=1}^K u'_{2,j} \exp(\iota 2\pi i t_{2,j}) + \eta'_{2,i}. \tag{3.14}
\end{aligned}$$

Here, $\eta'_{2,i} \in \mathbb{C}$ corresponds to noise arising from the estimation errors for the parameters in the first group of sources. As a direct consequence of Proposition 3, we have for each $j = 1, \dots, K$ that

$$\begin{aligned}
&|u_{1,j} \exp(\iota 2\pi(s_2 + i)t_{1,j}) - \widehat{u}_{1,\phi_1(j)} \exp(\iota 2\pi(s_2 + i)\widehat{t}_{1,\phi_1(j)})| \\
&\leq 2\pi u_{\max} |s_2 + i| d_w(t_{1,j}, \widehat{t}_{1,\phi_1(j)}) + |u_{1,j} - \widehat{u}_{1,\phi_1(j)}| \\
&< 2\pi u_{\max} |s_2 + i| \varepsilon_1 + \left(\bar{C}_1 + \bar{C}_2 \log^{1/2} \left(\frac{\bar{C}_3}{\varepsilon_1} \right) \right) \varepsilon_1, \tag{3.15}
\end{aligned}$$

where the last inequality above follows from the bounds on $|u_{1,j} - \widehat{u}_{1,\phi_1(j)}|$, $d_w(t_{1,j}, \widehat{t}_{1,\phi_1(j)})$, derived earlier. Now for $s_2 = 0$, and using the fact $|i| \leq m_2 < \frac{2}{\Delta_2(1-c)} + 1$ ($= M_{2,\text{up}}$), we obtain from (3.15) the following uniform bound on $|\eta'_{2,i}|$.

$$\begin{aligned}
|\eta'_{2,i}| &< \frac{\mu_1}{\mu_2} K e^{(2\pi^2 m_2^2 (\mu_2^2 - \mu_1^2))} \left(2\pi u_{\max} m_2 + \bar{C}_1 + \bar{C}_2 \log^{1/2} \left(\frac{\bar{C}_3}{\varepsilon_1} \right) \right) \varepsilon_1 \\
&< \frac{\mu_1}{\mu_2} K e^{(2\pi^2 M_{2,\text{up}}^2 (\mu_2^2 - \mu_1^2))} \left(2\pi u_{\max} M_{2,\text{up}} + \bar{C}_1 + \bar{C}_2 \log^{1/2} \left(\frac{\bar{C}_3}{\varepsilon_1} \right) \right) \varepsilon_1. \tag{3.16}
\end{aligned}$$

From (3.14), we see that $\widetilde{H}_0^{(2)} = V_2 D_{u'_2} V_2^H + E^{(2)}$, $\widetilde{H}_1^{(2)} = V_2 D_{u'_2} D_{\alpha_2} V_2^H + F^{(2)}$. Here, $D_{u'_2} = \text{diag}(u'_{2,1}, \dots, u'_{2,K})$ and $D_{\alpha_2} = \text{diag}(\alpha_{2,1}, \dots, \alpha_{2,K})$, while $E^{(2)}, F^{(2)}$ denote the perturbation matrices consisting of $(\eta'_{2,i})_{i=-m_2}^{m_2-1}$ terms, as in (2.2), (2.3).

We obtain the estimates $(\widehat{t}_{2,j})_{j=1}^K$ and $(\widehat{u}_{2,j})_{j=1}^K$ using the MMP method. Invoking Corollary 1 and assuming $\varepsilon_2 < c\Delta_2/2$, $\frac{2}{\Delta_2 - 2\varepsilon_2} + 1 \leq m_2 < M_{2,\text{up}}$ hold, it is sufficient that ε_1 satisfies the condition

$$\begin{aligned}
&\frac{\mu_1}{\mu_2} K e^{(2\pi^2 M_{2,\text{up}}^2 (\mu_2^2 - \mu_1^2))} \left(2\pi u_{\max} M_{2,\text{up}} + \bar{C}_1 + \bar{C}_2 \log^{1/2} \left(\frac{\bar{C}_3}{\varepsilon_1} \right) \right) \varepsilon_1 \\
&\leq \varepsilon_2 \frac{u_{\min}}{5C\sqrt{K}} \left(1 + 48 \frac{u_{\max}}{u_{\min}} \right)^{-1}.
\end{aligned}$$

Indeed, there then exists a permutation $\phi_2 : [K] \rightarrow [K]$ such that for each $j = 1, \dots, K$,

$$d_w(\widehat{t}_{2,\phi_2(j)}, t_{2,j}) \leq \varepsilon_2, \quad |\widehat{u}_{2,\phi_2(j)} - u_{2,j}| < \widetilde{C}_2 \varepsilon_2,$$

where $\widetilde{C}_2 = 4\pi K u_{\max} M_{2,\text{up}} + \frac{u_{\min}}{C\sqrt{K}} \left(1 + 16 \frac{u_{\max}^2}{u_{\min}^2}\right)^{-1}$. This completes the proof. \square

3.2 The general case

We now move to the general case where $L \geq 1$. The function f is now of the form

$$f(s) = \sum_{l=1}^L \bar{g}_l(s) \left(\sum_{j=1}^K u_{l,j} \exp(\iota 2\pi s t_{l,j}) \right)$$

where we recall that $\bar{g}_l(s) = \sqrt{2\pi} \mu_l \exp(-2\pi^2 s^2 \mu_l^2)$. Before stating our result, it will be helpful to define certain terms for ease of notation, later on.

(1) For $l = 1, \dots, L$,

$$M_{l,\text{up}} := \frac{2}{\Delta_l(1-c)} + 1, \quad \widetilde{C}_l := 4\pi K u_{\max} M_{l,\text{up}} + \frac{u_{\min}}{C\sqrt{K}} \left(1 + 16 \frac{u_{\max}^2}{u_{\min}^2}\right)^{-1}. \quad (3.17)$$

with constants $c \in (0, 1)$ and $C = 10 + \frac{1}{2\sqrt{2}}$ (from Theorem 3).

(2) For $l = 1, \dots, L-1$, and a constant $\widetilde{c} > 1$,

$$\bar{C}_{l,1} := \widetilde{C}_l + 2\pi u_{\max} \widetilde{c} M_{l,\text{up}}, \quad \bar{C}_{l,2} := \frac{2\pi u_{\max} \widetilde{c}}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}}, \quad (3.18)$$

$$D_l := \frac{5CK^{3/2}(L-l)u_{\max}\mu_L}{u_{\min}\mu_l} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right), \quad \bar{C}_{l,3} := \begin{cases} D_l & ; \quad l = 1 \\ 2D_l & ; \quad l > 1. \end{cases} \quad (3.19)$$

(3) For $l = 1, \dots, L$, define

$$E_l(\varepsilon) := \begin{cases} \left(\bar{C}_{l,1} + \bar{C}_{l,2} \log^{1/2} \left(\frac{\bar{C}_{l,3}}{\varepsilon} \right) \right) \varepsilon & ; \quad l < L \\ \bar{C}_L \varepsilon & ; \quad l = L. \end{cases} \quad (3.20)$$

Here, $\log^{1/2}(\cdot) := \sqrt{|\log(\cdot)|}$ and $\varepsilon_l \in (0, 1)$.

(4) For $l = 2, \dots, L-1$, define

$$F_l(\varepsilon) := C'_{l,1} + C'_{l,2} \log^{1/2} \left(\frac{2D_l}{\varepsilon} \right); \quad (3.21)$$

$$\text{where } C'_{l,1} := (\widetilde{c} + 1)M_{l,\text{up}}, \quad C'_{l,2} := \frac{\widetilde{c}}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}}. \quad (3.22)$$

Here, $\widetilde{c} > 1$ is the same constant as in (2).

We are now ready to state our main theorem for approximate recovery of the source parameters for each group.

Theorem 5. For a constant $c \in (0, 1)$, let $0 < \varepsilon_L < c\Delta_L/2$, $\frac{2}{\Delta_L - 2\varepsilon_L} \leq m_L < M_{L,up}$ and $s_L = 0$. Moreover, for $l = L - 1, \dots, 1$, say we choose ε_l, m_l, s_l as follows.

1. $\varepsilon_l < c\Delta_l/2$ additionally satisfies the following conditions.

(a) If $l > 1$, then $\varepsilon_l < D_l$.

(b) $2\pi u_{max} M_{L,up} \varepsilon_{L-1} + E_{L-1}(\varepsilon_{L-1}) \leq \varepsilon_L \frac{u_{min} \mu_L e^{-2\pi^2(\mu_L^2 - \mu_1^2)M_{L,up}^2}}{5CK^{3/2}(L-1)\mu_{L-1}} \left(1 + \frac{48u_{max}}{u_{min}}\right)^{-1}$.

(c) If $l < L - 1$, then $\varepsilon_l \leq \varepsilon_{l+1}$, $E_l(\varepsilon_l) \leq E_{l+1}(\varepsilon_{l+1})$ and

$$2\pi u_{max} F_{l+1}(\varepsilon_l) \varepsilon_l + E_l(\varepsilon_l) \leq \varepsilon_{l+1} e^{-2\pi^2(\mu_{l+1}^2 - \mu_1^2)F_{l+1}^2(\varepsilon_{l+1})} \frac{u_{min} \mu_{l+1}}{10CK^{3/2}l\mu_l} \left(1 + \frac{48u_{max}}{u_{min}}\right)^{-1}. \quad (3.23)$$

2. $\frac{2}{\Delta_l - 2\varepsilon_l} \leq m_l < M_{l,up}$, and $S_l \leq s_l \leq \tilde{c}S_l$ (for constant $\tilde{c} > 1$) where

$$S_l = m_l + \frac{1}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}} \log^{1/2} \left(\frac{b_l CK^{3/2} u_{max} (L-l) \mu_L}{\varepsilon_l u_{min} \mu_l} \left(1 + \frac{48u_{max}}{u_{min}}\right) \right),$$

with $b_l = 5$ if $l = 1$, and $b_l = 10$ otherwise.

Then, for each $l = 1, \dots, L$, there exists a permutation $\phi_l : [K] \rightarrow [K]$ such that

$$d_w(\hat{t}_{l,\phi_l(j)}, t_{l,j}) \leq \varepsilon_l, \quad |\hat{u}_{l,\phi_l(j)} - u_{l,j}| < E_l(\varepsilon_l); \quad j = 1, \dots, K.$$

Interpreting Theorem 5. Before proceeding to the proof, we make some useful observations.

- (a) We first choose the sampling parameters (ε, m, s) for the outermost kernel \bar{g}_L , then for \bar{g}_{L-1} , and so on. For the l^{th} group ($1 \leq l \leq L$), we first choose ε_L (accuracy), then m_l (number of samples), and finally s_l (sampling offset).
- (b) The choice of $\varepsilon_L \in (0, c\Delta_L/2)$, while free, dictates the choice of $\varepsilon_1, \dots, \varepsilon_{L-1}$. To begin with, condition 1b essentially requires ε_{L-1} to be sufficiently small with respect to ε_L . Similarly, for $l = 1, \dots, L-2$, the conditions in 1c require ε_l to be sufficiently small with respect to ε_{l+1} . It ensures that during the estimation of the parameters for the $(l+1)^{th}$ group, the estimation errors carrying forward from the previous groups (1 to l) are sufficiently small.
- (c) For each $l (< L)$, the lower bound on s_l is to ensure that we are sufficiently deep in the tails of $\bar{g}_{l+1}, \bar{g}_{l+2}, \dots, \bar{g}_L$. The upper bound on s_l is to control the estimation errors of the source amplitudes for group l (see (2.9)).

Order wise dependencies. We now discussing the scaling of the terms involved, assuming $u_{max}, u_{min} = \Theta(1)$.

- (i) For $l = 1, \dots, L$, we have $M_{l,up} = \Theta(\frac{1}{\Delta_l})$, $\tilde{C}_l = \Theta(\frac{K}{\Delta_l})$.
- (ii) For $p = 1, \dots, L-1$ we have

$$\bar{C}_{p,1} = \Theta\left(\frac{K}{\Delta_p}\right), \quad \bar{C}_{p,2} = \Theta\left(\frac{1}{(\mu_{p+1}^2 - \mu_p^2)^{1/2}}\right), \quad D_p = \Theta\left(\frac{K^{3/2}(L-p)\mu_L}{\mu_p}\right), \quad \bar{C}_{p,3} = \Theta(D_p).$$

(iii) For $l = 1, \dots, L$, we have

$$E_l(\varepsilon_l) := \begin{cases} \Theta \left(\left(\frac{K}{\Delta_l} + \frac{1}{(\mu_{l+1}^2 - \mu_l^2)^{1/2}} \log^{1/2} \left(\frac{K^{3/2}(L-l)\mu_L}{\varepsilon_l \mu_l} \right) \right) \varepsilon_l \right) & ; \quad l < L \\ \Theta(\frac{K}{\Delta_L})\varepsilon_L & ; \quad l = L. \end{cases} \quad (3.24)$$

(iv) For $q = 2, \dots, L-1$, we have

$$F_q(\varepsilon_q) = \Theta \left(\frac{1}{\Delta_q} + \frac{1}{(\mu_{q+1}^2 - \mu_q^2)^{1/2}} \log^{1/2} \left(\frac{K^{3/2}(L-q)\mu_L}{\varepsilon_q \mu_q} \right) \right). \quad (3.25)$$

Conditions on ε_l . Theorem 5 has several conditions on ε_l , which might be difficult to digest at first glance. On a top level, the conditions dictate that the accuracies satisfy $\varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_L$. In fact, they require a stronger condition in the sense that for each $1 \leq l \leq L-1$, ε_l is required to be sufficiently smaller than ε_{l+1} (the choice of $\varepsilon_L \lesssim \Delta_L/2$ is free). This places the strongest assumption on ε_1 meaning that the source parameters corresponding to the “outermost kernel” in the Fourier domain should be estimated with the highest accuracy. Below, we state the conditions appearing on ε_l in the Theorem up to positive constants; the details are deferred to Appendix C.

1. Condition 1b in Theorem 5 holds if $\varepsilon_{L-1}, \varepsilon_L$ satisfy

$$\varepsilon_{L-1} \lesssim \alpha(\varepsilon_L)^{\frac{1}{1-\theta}} \quad (3.26)$$

for any given $\theta \in (0, 1/2)$. Here, $\alpha > 0$ depends on $\theta, K, L, \Delta_{L-1}, \Delta_L, \mu_L, \mu_{L-1}, \mu_1$.

2. For $l < L-1$, let us look at condition 1c in Theorem 5. The requirement $E_l(\varepsilon_l) \leq E_{l+1}(\varepsilon_{l+1})$ holds if $\varepsilon_l, \varepsilon_{l+1}$ satisfy

$$\varepsilon_l \lesssim \lambda_l \log^{\frac{1}{2(1-\theta)}} \left(\frac{K^{3/2}\mu_L L}{\varepsilon_{l+1}\mu_{l+1}} \right) \varepsilon_{l+1}^{\frac{1}{1-\theta}} \quad (3.27)$$

for any given $\theta \in (0, 1/2)$. Here, $\lambda_l > 0$ depends on $\Delta_l, \Delta_{l+1}, \mu_l, \mu_{l+1}, \mu_{l+2}, L, K, \theta$.

Furthermore, the condition in (3.23) is satisfied if $\varepsilon_l, \varepsilon_{l+1}$ satisfy

$$\varepsilon_l \lesssim \beta_l(\varepsilon_{l+1})^{\frac{1+\gamma_l}{1-\theta}} \quad (3.28)$$

for any given $\theta \in (0, 1/2)$. Here, $\beta_l > 0$ depends on $L, K, \Delta_l, \Delta_{l+1}, \Delta_{L-1}, \Delta_L, \mu_l, \mu_{l+1}, \mu_{l+2}, \mu_L, \mu_{L-1}, \mu_1, \theta$, while $\gamma_l > 0$ depends on $\mu_1, \mu_{l+1}, \mu_{l+2}, \Delta_{l+1}$. Note that the dependence on ε_{l+1} is stricter in (3.28) as compared to (3.27).

Effect of separation between μ_l, μ_{l+1} for $l = 1, \dots, L-1$. The interaction between μ_{L-1}, μ_L occurs in the same manner as explained for the case two kernel case, the reader is invited to verify this. We analyze the interaction between μ_l, μ_{l+1} below for $l < L-1$.

1. Consider the scenario where $\mu_l \rightarrow \mu_{l+1}$ (with other terms fixed). We see that s_l has to be suitably large now in order to be able to distinguish between \bar{g}_l and \bar{g}_{l+1} . Moreover, conditions (3.27), (3.28) become stricter in the sense that $\lambda_l, \beta_l \rightarrow 0$.

2. Now say μ_{l+1} is fixed, and $\mu_l \rightarrow 0$ (and hence $\mu_1, \dots, \mu_{l-1} \rightarrow 0$). In this case, the conditions $E_1(\varepsilon_1) \leq \dots \leq E_{l+1}(\varepsilon_{l+1})$ become vacuous as the estimation error arising from stages $1, \dots, l-1$ themselves approach 0. However, s_1, \dots, s_l now increase accordingly in order to distinguish within $\bar{g}_1, \dots, \bar{g}_l$. Accordingly, to control the estimation error of the amplitudes, i.e, $E_i(\varepsilon_i)$; $1 \leq i \leq l$, $\varepsilon_1, \dots, \varepsilon_l$ have to be suitably small.

Proof of Theorem 5. The proof is divided in to three main steps.

- **Recovering source parameters for first group.** For $i = -m_1, \dots, m_1 - 1$, we have

$$\begin{aligned}
\frac{f(s_1 + i)}{\bar{g}_1(s_1 + i)} &= \sum_{j=1}^K u_{1,j} \exp(\iota 2\pi(s_1 + i)t_{1,j}) + \sum_{l=2}^L \frac{\bar{g}_l(s_1 + i)}{\bar{g}_1(s_1 + i)} \sum_{j=1}^K u_{l,j} \exp(\iota 2\pi(s_1 + i)t_{l,j}) \\
&= \sum_{j=1}^K \underbrace{u_{1,j} \exp(\iota 2\pi s_1 t_{1,j})}_{u'_{1,j}} \exp(\iota 2\pi i t_{1,j}) \\
&\quad + \underbrace{\sum_{l=2}^L \frac{\mu_l}{\mu_1} \exp(-2\pi^2(s_1 + i)^2(\mu_l^2 - \mu_1^2)) \sum_{j=1}^K u_{l,j} \exp(\iota 2\pi(s_1 + i)t_{l,j})}_{\eta'_{1,i}} \\
&= \sum_{j=1}^K u'_{1,j} \exp(\iota 2\pi i t_{1,j}) + \eta'_{1,i}. \tag{3.29}
\end{aligned}$$

Here, $\eta'_{1,i}$ is the perturbation due to the tail of $\bar{g}_2, \bar{g}_3, \dots, \bar{g}_L$. Since the stated choice of s_1 implies $s_1 > m_1$, this means $\min_i (s_1 + i)^2 = (s_1 - m_1)^2$, and hence clearly

$$\begin{aligned}
|\eta'_{1,i}| &\leq K u_{\max} \sum_{l=2}^L \frac{\mu_l}{\mu_1} \exp(-2\pi^2(s_1 - m_1)^2(\mu_l^2 - \mu_1^2)) \\
&\leq K u_{\max}(L-1) \frac{\mu_L}{\mu_1} \exp(-2\pi^2(s_1 - m_1)^2(\mu_2^2 - \mu_1^2))
\end{aligned}$$

We obtain estimates $\hat{t}_{1,j}, \hat{u}_{1,j}, j = 1, \dots, K$ via the MMP method. Invoking Corollary 1, we have for $\varepsilon_1 < c\Delta_1/2$, and $\frac{2}{\Delta_1 - 2\varepsilon_1} + 1 \leq m_1 < \frac{2}{\Delta_1(1-c)} + 1 (= M_{1,\text{up}})$ that if s_1 satisfies

$$K u_{\max}(L-1) \frac{\mu_L}{\mu_1} \exp(-2\pi^2(s_1 - m_1)^2(\mu_2^2 - \mu_1^2)) \leq \varepsilon_1 \frac{u_{\min}}{5C\sqrt{K}} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right)^{-1} \tag{3.30}$$

then there exists a permutation $\phi_1 : [K] \rightarrow [K]$ such that for each $j = 1, \dots, K$,

$$d_w(\hat{t}_{1,\phi_1(j)}, t_{1,j}) \leq \varepsilon_1, \quad |\hat{u}_{1,\phi_1(j)} - u_{1,j}| < (\tilde{C}_1 + 2\pi u_{\max} s_1) \varepsilon_1. \tag{3.31}$$

Clearly, the condition

$$s_1 \geq m_1 + \frac{1}{(2\pi^2(\mu_2^2 - \mu_1^2))^{1/2}} \log^{1/2} \left(\frac{5CK^{3/2}(L-1)u_{\max}\mu_L}{u_{\min}\mu_1\varepsilon_1} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right) \right) \tag{3.32}$$

implies (3.30). Moreover, since $s_1 \leq \tilde{c}S_1$ and $m_1 < M_{1,\text{up}}$, we obtain

$$s_1 < \tilde{c} \left(M_{1,\text{up}} + \frac{1}{(2\pi^2(\mu_2^2 - \mu_1^2))^{1/2}} \log^{1/2} \left(\frac{5CK^{3/2}(L-1)u_{\max}\mu_L}{u_{\min}\mu_1\varepsilon_1} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right) \right) \right). \tag{3.33}$$

Plugging (3.33) in (3.31), we obtain

$$|\widehat{u}_{1,\phi_1(j)} - u_{1,j}| < \left(\bar{C}_{1,1} + \bar{C}_{1,2} \log^{1/2} \left(\frac{\bar{C}_{1,3}}{\varepsilon_1} \right) \right) \varepsilon_1 = E_1(\varepsilon_1); \quad j = 1, \dots, K, \quad (3.34)$$

where $\bar{C}_{1,1}, \bar{C}_{1,2}, \bar{C}_{1,3} > 0$ are constants defined in (3.18), (3.19), and $E_p(\cdot)$ is defined in (3.20).

- **Recovering source parameters for l^{th} ($1 < l < L$) group.** Say we are at the l^{th} iteration for $1 < l < L$, having estimated the source parameters up to the $(l-1)^{th}$ group. Say that for each $p = 1, \dots, l-1$ and $j = 1, \dots, K$ the following holds.

$$d_w(\widehat{t}_{p,\phi_p(j)}, t_{p,j}) \leq \varepsilon_p, \quad |\widehat{u}_{p,\phi_p(j)} - u_{p,j}| < E_p(\varepsilon_p), \quad (3.35)$$

for some permutations $\phi_p : [K] \rightarrow [K]$, with

1. $\varepsilon_1 \leq \dots \leq \varepsilon_{l-1}$; $E_1(\varepsilon_1) \leq \dots \leq E_{l-1}(\varepsilon_{l-1})$;
2. $\varepsilon_p < c\Delta_p/2$; $\varepsilon_q < D_q$, $2 \leq q \leq l-1$;
3. $2\pi u_{\max} F_{q+1}(\varepsilon_q) \varepsilon_q + E_q(\varepsilon_q) \leq \varepsilon_{q+1} e^{-2\pi^2(\mu_{q+1}^2 - \mu_1^2) F_{q+1}^2(\varepsilon_{q+1})} \frac{u_{\min} \mu_{q+1}}{10CK^{3/2}q\mu_q} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1}$, $1 \leq q \leq l-2$.

For $i = -m_l, \dots, m_l - 1$, we have

$$\begin{aligned} & \frac{f(s_l + i) - \sum_{p=1}^{l-1} \widehat{f}_p(s_l + i)}{\bar{g}_l(s_l + i)} \\ &= \underbrace{\sum_{p=1}^{l-1} \frac{\bar{g}_p(s_l + i)}{\bar{g}_l(s_l + i)} \sum_{j=1}^K [u_{p,j} \exp(\iota 2\pi(s_l + i)t_{p,j}) - \widehat{u}_{p,\phi_p(j)} \exp(\iota 2\pi(s_l + i)\widehat{t}_{p,\phi_p(j)})]}_{\eta'_{l,i,past}} \\ &+ \underbrace{\sum_{q=l+1}^L \frac{\bar{g}_q(s_l + i)}{\bar{g}_l(s_l + i)} \sum_{j=1}^K u_{q,j} \exp(\iota 2\pi(s_l + i)t_{q,j})}_{\eta'_{l,i,fut}} \\ &+ \sum_{j=1}^K \underbrace{u_{l,j} \exp(\iota 2\pi s_l t_{l,j}) \exp(\iota 2\pi i t_{l,j})}_{u'_{l,j}} = \sum_{j=1}^K u'_{l,j} \exp(\iota 2\pi i t_{l,j}) + \eta'_{l,i,past} + \eta'_{l,i,fut}. \end{aligned}$$

Here, $\eta'_{l,i,past}$ denotes perturbation due to the estimation errors of the source parameters in the past. Moreover, $\eta'_{l,i,fut}$ denotes perturbation due to the tails of the kernels that are yet to be processed.

- (i) Bounding $\eta'_{l,i,past}$. To begin with, note

$$\begin{aligned} \eta'_{l,i,past} &= \sum_{p=1}^{l-1} \frac{\mu_p}{\mu_l} \exp(2\pi^2(\mu_l^2 - \mu_p^2)(s_l + i)^2) \sum_{j=1}^K [u_{p,j} \exp(\iota 2\pi(s_l + i)t_{p,j}) \\ &\quad - \widehat{u}_{p,\phi_p(j)} \exp(\iota 2\pi(s_l + i)\widehat{t}_{p,\phi_p(j)})]. \end{aligned}$$

Using Proposition 3, we have for each $p = 1, \dots, l-1$ and $j = 1, \dots, K$ that

$$\begin{aligned} & |u_{p,j} \exp(\iota 2\pi(s_l + i)t_{p,j}) - \widehat{u}_{p,\phi_p(j)} \exp(\iota 2\pi(s_l + i)\widehat{t}_{p,\phi_p(j)})| \\ & \leq 2\pi u_{\max} |s_l + i| d_w(t_{p,j}, \widehat{t}_{p,\phi_p(j)}) + |u_{p,j} - \widehat{u}_{p,\phi_p(j)}| \\ & < 2\pi u_{\max} |s_l + i| \varepsilon_p + E_p(\varepsilon_p), \end{aligned} \quad (3.36)$$

where the last inequality is due to (3.35). Since $s_l > m_l$, hence $(s_l + i)^2 < (s_l + m_l)^2$ for all $i = -m_l, \dots, m_l - 1$. With the help of (3.36), we then readily obtain

$$\begin{aligned} |\eta'_{l,i,past}| & < \sum_{p=1}^{l-1} \left(\left(\frac{\mu_p}{\mu_l} \exp(2\pi^2(\mu_l^2 - \mu_p^2)(s_l + m_l)^2) \right) (2\pi u_{\max}(s_l + m_l)\varepsilon_p + E_p(\varepsilon_p))K \right) \\ & \leq \left(K \frac{\mu_{l-1}}{\mu_l} e^{2\pi^2(\mu_l^2 - \mu_1^2)(s_l + m_l)^2} \right) \left(\sum_{p=1}^{l-1} 2\pi u_{\max}(s_l + m_l)\varepsilon_p + E_p(\varepsilon_p) \right) \\ & \leq \left(K(l-1) \frac{\mu_{l-1}}{\mu_l} e^{2\pi^2(\mu_l^2 - \mu_1^2)(s_l + m_l)^2} \right) (2\pi u_{\max}(s_l + m_l)\varepsilon_{l-1} + E_{l-1}(\varepsilon_{l-1})), \end{aligned} \quad (3.37)$$

where in the last inequality, we used 1.

(ii) Bounding $\eta'_{l,i,fut}$. We have

$$\eta'_{l,i,fut} = \sum_{q=l+1}^L \frac{\mu_q}{\mu_l} \exp(-2\pi^2(s_l + i)^2(\mu_q^2 - \mu_l^2)) \left(\sum_{j=1}^K u_{q,j} \exp(\iota 2\pi(s_l + i)t_{q,j}) \right).$$

Since $s_l > m_l$, we have $(s_l + i)^2 \geq (s_l - m_l)^2$ for all $i = -m_l, \dots, m_l - 1$. This, along with the fact $\frac{\mu_q}{\mu_l} \leq \frac{\mu_L}{\mu_l}$ gives us

$$|\eta'_{l,i,fut}| \leq \frac{\mu_L}{\mu_l} K u_{\max}(L - l) \exp(-2\pi^2(s_l - m_l)^2(\mu_{l+1}^2 - \mu_l^2)).$$

It follows that if $s_l \geq S_l$ where

$$S_l = m_l + \frac{1}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}} \log^{1/2} \left(\frac{10CK^{3/2}u_{\max}(L-l)\mu_L}{\varepsilon_l u_{\min}\mu_l} \left(1 + \frac{48u_{\max}}{u_{\min}} \right) \right)$$

then for $i = -m_l, \dots, m_l - 1$, we have

$$|\eta'_{l,i,fut}| < \varepsilon_l \frac{u_{\min}}{10C\sqrt{K}} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1}. \quad (3.38)$$

(iii) Back to $\eta'_{l,i,past}$. We will now find conditions which ensure that the same bound as (3.38) holds on $|\eta'_{l,i,past}|$, uniformly for all i . To this end, since $s_l \leq \widetilde{c}S_l$ and $m_l < M_{l,\text{up}}$, we obtain

$$s_l < \widetilde{c} \left(M_{l,\text{up}} + \frac{1}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}} \log^{1/2} \left(\frac{10CK^{3/2}(L-l)u_{\max}\mu_L}{u_{\min}\mu_l\varepsilon_l} \left(1 + 48\frac{u_{\max}}{u_{\min}} \right) \right) \right). \quad (3.39)$$

This then gives us the bound

$$\begin{aligned}
s_l + m_l &< (\tilde{c} + 1)M_{l,\text{up}} + \frac{\tilde{c}}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}} \\
&\quad \cdot \log^{1/2} \left(\frac{10CK^{3/2}(L-l)u_{\max}\mu_L}{u_{\min}\mu_l\varepsilon_l} \left(1 + 48\frac{u_{\max}}{u_{\min}} \right) \right). \\
&= C'_{l,1} + C'_{l,2} \log^{1/2} \left(\frac{2D_l}{\varepsilon_l} \right) = F_l(\varepsilon_l),
\end{aligned}$$

where we recall the definition of F_l , and constants $C'_{l,1}, C'_{l,2}, D_l > 0$ from (3.21), (3.22). If $\varepsilon_l < D_l$ and $\varepsilon_l \geq \varepsilon_{l-1}$ hold, then $(s_l + m_l) < F_l(\varepsilon_l) \leq F_l(\varepsilon_{l-1})$. Using this in (3.37), we obtain

$$|\eta'_{l,i,\text{past}}| < \left(K(l-1) \frac{\mu_{l-1}}{\mu_l} e^{2\pi^2(\mu_l^2 - \mu_1^2)F_l^2(\varepsilon_l)} \right) (2\pi u_{\max} F_l(\varepsilon_{l-1})\varepsilon_{l-1} + E_{l-1}(\varepsilon_{l-1})) \quad (3.40)$$

Therefore if ε_{l-1} satisfies the condition

$$\begin{aligned}
&2\pi u_{\max} F_l(\varepsilon_{l-1})\varepsilon_{l-1} + E_{l-1}(\varepsilon_{l-1}) \\
&\leq \varepsilon_l e^{-2\pi^2(\mu_l^2 - \mu_1^2)F_l^2(\varepsilon_l)} \frac{u_{\min}\mu_l}{10CK^{3/2}(l-1)\mu_{l-1}} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1}
\end{aligned}$$

then it implies $|\eta'_{l,i,\text{past}}| < \varepsilon_l \frac{u_{\min}}{10C\sqrt{K}} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1}$. Together with (3.38), this gives

$$|\eta'_{l,i}| \leq |\eta'_{l,i,\text{fut}}| + |\eta'_{l,i,\text{past}}| < \varepsilon_l \frac{u_{\min}}{5C\sqrt{K}} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1}.$$

We obtain estimates $\hat{t}_{l,j}, \hat{u}_{l,j}, j = 1, \dots, K$ via the MMP method. Invoking Corollary 1, if $\varepsilon_l < c\Delta_l/2$, then for the stated choice of m_l , there exists a permutation $\phi_l : [K] \rightarrow [K]$ such that for each $j = 1, \dots, K$,

$$d_w(\hat{t}_{l,\phi_l(j)}, t_{l,j}) \leq \varepsilon_l; \quad |\hat{u}_{l,\phi_l(j)} - u_{l,j}| < \left(\tilde{C}_l + 2\pi u_{\max} s_l \right) \varepsilon_l < E_l(\varepsilon_l). \quad (3.41)$$

The last inequality follows readily using (3.39).

- **Recovering source parameters for last group.** Say that for each $p = 1, \dots, L-1$ and $j = 1, \dots, K$ the following holds.

$$d_w(\hat{t}_{p,\phi_p(j)}, t_{p,j}) \leq \varepsilon_p, \quad |\hat{u}_{p,\phi_p(j)} - u_{p,j}| < E_p(\varepsilon_p), \quad (3.42)$$

for some permutations $\phi_p : [K] \rightarrow [K]$, with

1. $\varepsilon_1 \leq \dots \leq \varepsilon_{L-1}; E_1(\varepsilon_1) \leq \dots \leq E_{L-1}(\varepsilon_{L-1});$
2. $\varepsilon_p < c\Delta_p/2; \varepsilon_q < D_q, 2 \leq q \leq L-1;$
3. $2\pi u_{\max} F_{q+1}(\varepsilon_q)\varepsilon_q + E_q(\varepsilon_q) \leq \varepsilon_{q+1} e^{-2\pi^2(\mu_{q+1}^2 - \mu_1^2)F_{q+1}^2(\varepsilon_{q+1})} \frac{u_{\min}\mu_{q+1}}{10CK^{3/2}q\mu_q} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1}, 1 \leq q \leq L-2.$

We proceed by noting that for each $i = -m_L, \dots, m_L - 1$

$$\begin{aligned}
& \frac{f(s_L + i) - \sum_{p=1}^{L-1} \widehat{f}_p(s_L + i)}{\bar{g}_L(s_L + i)} \\
&= \underbrace{\sum_{p=1}^{L-1} \frac{\bar{g}_p(s_L + i)}{\bar{g}_L(s_L + i)} \sum_{j=1}^K [u_{p,j} \exp(\iota 2\pi(s_L + i)t_{p,j}) - \widehat{u}_{p,\phi_p(j)} \exp(\iota 2\pi(s_L + i)\widehat{t}_{p,\phi_p(j)})]}_{\eta'_{L,i}} \\
&+ \sum_{j=1}^K \underbrace{u_{L,j} \exp(\iota 2\pi s_L t_{L,j})}_{u'_{L,j}} \exp(\iota 2\pi i t_{L,j}) = \sum_{j=1}^K u'_{L,j} \exp(\iota 2\pi i t_{L,j}) + \eta'_{L,i}.
\end{aligned}$$

Using Proposition 3, we have for each $p = 1, \dots, L-1$ and $j = 1, \dots, K$ that

$$\begin{aligned}
& |u_{p,j} \exp(\iota 2\pi(s_L + i)t_{p,j}) - \widehat{u}_{p,\phi_p(j)} \exp(\iota 2\pi(s_L + i)\widehat{t}_{p,\phi_p(j)})| \\
&\leq 2\pi u_{\max} |s_L + i| d_w(t_{p,j}, \widehat{t}_{p,\phi_p(j)}) + |u_{p,j} - \widehat{u}_{p,\phi_p(j)}| \\
&< 2\pi u_{\max} |s_L + i| \varepsilon_p + E_p(\varepsilon_p),
\end{aligned} \tag{3.43}$$

where the last inequality follows from (3.42). Since $s_L = 0$, hence $(s_L + i)^2 < m_L^2 < M_{L,\text{up}}^2$ for all $i = -m_L, \dots, m_L - 1$. Using (3.43), we then readily obtain

$$\begin{aligned}
|\eta'_{L,i}| &< \sum_{p=1}^{L-1} \left(\left(\frac{\mu_p}{\mu_L} e^{2\pi^2(\mu_L^2 - \mu_p^2)M_{L,\text{up}}^2} \right) (2\pi u_{\max} M_{L,\text{up}} \varepsilon_p + E_p(\varepsilon_p)) K \right) \\
&\leq \left(K \frac{\mu_{L-1}}{\mu_L} e^{2\pi^2(\mu_L^2 - \mu_1^2)M_{L,\text{up}}^2} \right) \left(\sum_{p=1}^{L-1} 2\pi u_{\max} M_{L,\text{up}} \varepsilon_p + E_p(\varepsilon_p) \right) \\
&\leq \left(K(L-1) \frac{\mu_{L-1}}{\mu_L} e^{2\pi^2(\mu_L^2 - \mu_1^2)M_{L,\text{up}}^2} \right) (2\pi u_{\max} M_{L,\text{up}} \varepsilon_{L-1} + E_{L-1}(\varepsilon_{L-1})) \tag{3.44}
\end{aligned}$$

where in the last inequality, we used (1).

We obtain estimates $\widehat{t}_{L,j}, \widehat{u}_{L,j}, j = 1, \dots, K$ via the MMP method. Invoking Corollary 1 and assuming $\varepsilon_L < c\Delta_L/2$, it follows for the stated conditions on m_L , that it suffices if ε_{L-1} satisfies

$$\begin{aligned}
& 2\pi u_{\max} M_{L,\text{up}} \varepsilon_{L-1} + E_{L-1}(\varepsilon_{L-1}) \\
&\leq \varepsilon_L \frac{e^{-2\pi^2(\mu_L^2 - \mu_1^2)M_{L,\text{up}}^2} u_{\min} \mu_L}{5CK^{3/2}(L-1)\mu_{L-1}} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1}.
\end{aligned}$$

Indeed, there then exists a permutation $\phi_L : [K] \rightarrow [K]$ such that for each $j = 1, \dots, K$,

$$d_w(\widehat{t}_{L,\phi_L(j)}, t_{L,j}) \leq \varepsilon_L; \quad |\widehat{u}_{L,\phi_L(j)} - u_{L,j}| < \widetilde{C}_L \varepsilon_L = E_L(\varepsilon_L). \tag{3.45}$$

This completes the proof. □

4 Unmixing Gaussians in Fourier domain: Noisy case

We now analyze the noisy setting where we acquire noisy values of the of the Fourier transform of f at the sampling location (frequency) s . In particular, at stage p ($1 \leq p \leq L$) in Algorithm 2, and frequency s , let $w_p(s)$ denote the additive observation noise on the clean Fourier sample $f(s)$. Denoting the noisy measurement by $\tilde{f}(s)$, this means that at stage p ,

$$\begin{aligned}\tilde{f}(s) &= f(s) + w_p(s) \\ &= \sum_{l=1}^L \bar{g}_l(s) \left(\sum_{j=1}^K u_{l,j} \exp(\iota 2\pi s t_{l,j}) \right) + w_p(s).\end{aligned}$$

In addition to the terms defined at the beginning of Section 3.2, we will need an additional term (defined below) which will be used in the statement of our theorem.

$$F'_l(\varepsilon_l) := C'_{l,1} + C'_{l,2} \log^{1/2} \left(\frac{3D_l}{\varepsilon_l} \right); \quad l = 2, \dots, L-1, \quad (4.1)$$

where $C'_{l,1}$, $C'_{l,2}$ are as defined in (3.22).

Theorem 6. *For a constant $c \in (0, 1)$, let $0 < \varepsilon_L < c\Delta_L/2$, $\frac{2}{\Delta_L - 2\varepsilon_L} \leq m_L < M_{L,up}$ and $s_L = 0$. Moreover, for $l = L-1, \dots, 1$, say we choose ε_l, m_l, s_l as follows.*

1. $\varepsilon_l < c\Delta_l/2$ additionally satisfies the following conditions.

(a) If $l > 1$, then $\varepsilon_l < D_l$.

(b) $2\pi u_{max} M_{L,up} \varepsilon_{L-1} + E_{L-1}(\varepsilon_{L-1}) \leq \varepsilon_L \frac{u_{min} \mu_L e^{-2\pi^2(\mu_L^2 - \mu_1^2) M_{L,up}^2}}{10CK^{3/2}(L-1)\mu_{L-1}} \left(1 + \frac{48u_{max}}{u_{min}} \right)^{-1}$.

(c) If $l < L-1$, then $\varepsilon_l \leq \varepsilon_{l+1}$, $E_l(\varepsilon_l) \leq E_{l+1}(\varepsilon_{l+1})$ and

$$2\pi u_{max} F'_{l+1}(\varepsilon_l) \varepsilon_l + E_l(\varepsilon_l) \leq \varepsilon_{l+1} e^{-2\pi^2(\mu_{l+1}^2 - \mu_1^2) F_{l+1}'^2(\varepsilon_{l+1})} \frac{u_{min} \mu_{l+1}}{15CK^{3/2}l\mu_l} \left(1 + \frac{48u_{max}}{u_{min}} \right)^{-1}.$$

2. $\frac{2}{\Delta_l - 2\varepsilon_l} \leq m_l < M_{l,up}$, and $S_l \leq s_l \leq \tilde{c}S_l$ (for constant $\tilde{c} > 1$) where

$$S_l = m_l + \frac{1}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}} \log^{1/2} \left(\frac{b_l CK^{3/2} u_{max} (L-l) \mu_L}{\varepsilon_l u_{min} \mu_l} \left(1 + \frac{48u_{max}}{u_{min}} \right) \right),$$

with $b_l = 10$ if $l = 1$, and $b_l = 15$ otherwise.

Moreover, assume that the noise satisfies the conditions

$$\left\| \frac{w_1(\cdot)}{\bar{g}_1(s_1 + \cdot)} \right\|_{\infty} \leq \varepsilon_1 \frac{u_{min}}{10C\sqrt{K}} \left(1 + 48 \frac{u_{max}}{u_{min}} \right)^{-1}, \quad (4.2)$$

$$\left\| \frac{w_l(\cdot)}{\bar{g}_l(s_l + \cdot)} \right\|_{\infty} \leq \varepsilon_l \frac{u_{min}}{15C\sqrt{K}} \left(1 + 48 \frac{u_{max}}{u_{min}} \right)^{-1}, \quad l = 2, \dots, L-1, \quad \text{and} \quad (4.3)$$

$$\left\| \frac{w_L(\cdot)}{\bar{g}_L(s_L + \cdot)} \right\|_{\infty} \leq \varepsilon_L \frac{u_{min}}{10C\sqrt{K}} \left(1 + 48 \frac{u_{max}}{u_{min}} \right)^{-1}. \quad (4.4)$$

Then, for each $l = 1, \dots, L$, there exists a permutation $\phi_l : [K] \rightarrow [K]$ such that

$$d_w(\hat{t}_{l,\phi_l(j)}, t_{l,j}) \leq \varepsilon_l, \quad |\hat{u}_{l,\phi_l(j)} - u_{l,j}| < E_l(\varepsilon_l); \quad j = 1, \dots, K.$$

Since the organisation and the arguments of the proof of Theorem 6 are almost identical to that of Theorem 5, we defer this proof to Appendix D where we pinpoint the main differences between the noiseless and the noisy settings.

Interpreting Theorem 6 Theorem 6 is almost the same as Theorem 5 barring the conditions on the magnitude of external noise (and minor differences in some constants). Specifically, conditions (4.2) - (4.4) state that at stage l , the magnitude of the noise should be small relative to the desired accuracy parameter ε_l . This is examined in more detail below. For convenience, we will now assume $u_{\min}, u_{\max} = \Theta(1)$.

1. **(Condition on $\|w_l(\cdot)\|_\infty$ for $1 \leq l \leq L-1$.)** Let us start with the case $2 \leq l \leq L-1$. Condition (4.3) states that

$$\left| \frac{w_l(i)}{\bar{g}_l(s_l+i)} \right| \lesssim \frac{\varepsilon_l}{\sqrt{K}}; \quad i = -m_l, \dots, m_l - 1. \quad (4.5)$$

Now, $\left| \frac{w_l(i)}{\bar{g}_l(s_l+i)} \right| = \frac{|w_l(i)|}{\sqrt{2\pi\mu_l}} e^{2\pi^2(s_l+i)^2\mu_l^2}$. Since $s_l > m_l$, therefore $s_l + i > 0$ for the given range of i . Hence, $(s_l + i)^2 < (s_l + m_l)^2 < F_l^2(\varepsilon)$ for each i . Using the order wise dependency from (3.25), we obtain for each i that

$$\frac{|w_l(i)|}{\sqrt{2\pi\mu_l}} e^{2\pi^2(s_l+i)^2\mu_l^2} \lesssim \frac{|w_l(i)|}{\mu_l} \left(\frac{K^{3/2}L\mu_L}{\varepsilon_l\mu_l} \right)^{C(\mu_l, \mu_{l+1}, \Delta_l)}, \quad (4.6)$$

where $C(\mu_l, \mu_{l+1}, \Delta_l) > 0$ depends only on $\mu_l, \mu_{l+1}, \Delta_l$. Hence from (4.5), (4.6), we see that (4.3) is satisfied if

$$\|w_l\|_\infty \lesssim \frac{(\varepsilon_l\mu_l)^{1+C(\mu_l, \mu_{l+1}, \Delta_l)}}{\sqrt{K}(K^{3/2}L\mu_L)^{C(\mu_l, \mu_{l+1}, \Delta_l)}}; \quad 2 \leq l \leq L-1. \quad (4.7)$$

In a similar manner, one can easily show that (4.2) is satisfied if

$$\|w_1\|_\infty \lesssim \frac{(\varepsilon_1\mu_1)^{1+C(\mu_1, \mu_2, \Delta_1)}}{\sqrt{K}(K^{3/2}L\mu_L)^{C(\mu_1, \mu_2, \Delta_1)}}. \quad (4.8)$$

2. **(Condition on $\|w_L\|_\infty$.)** In this case, $s_L = 0$ and so $(s_L + i)^2 \leq m_L^2$. Therefore for each $i = -m_L, \dots, m_L - 1$, we obtain

$$\left| \frac{w_L(i)}{\bar{g}_L(s_L+i)} \right| = \frac{|w_L(i)|}{\sqrt{2\pi\mu_L}} e^{2\pi^2(i)^2\mu_L^2} \lesssim \frac{|w_L(i)|}{\mu_L} e^{2\pi^2\mu_L^2/\Delta_L^2}. \quad (4.9)$$

Hence from (4.9), we see that (4.4) is satisfied if

$$\|w_L\|_\infty \lesssim \mu_L e^{-\mu_L^2/\Delta_L^2} \frac{\varepsilon_L}{\sqrt{K}}. \quad (4.10)$$

(4.8), (4.7), (4.10) show the conditions that the noise level is required to satisfy at the different levels. From the discussion following Theorem 5, we know that the ε_i 's gradually become smaller and smaller as we move from $i = L$ to $i = 1$ (with ε_1 being the smallest). Therefore the condition on $\|w_1\|_\infty$ is the strictest, while the condition on $\|w_L\|_\infty$ is the mildest.

Corollary for the case $L = 1$. As noted earlier, the case $L = 1$ is not interesting in the absence of external noise as we can exactly recover the source parameters. The situation is more interesting in the presence of noise as shown in the following Corollary of Theorem 6 for the case $L = 1$.

Corollary 2. For a constant $c \in (0, 1)$, let $0 < \varepsilon_1 < c\Delta_1/2$, $\frac{2}{\Delta_1 - 2\varepsilon_1} \leq m_1 < \frac{2}{\Delta_1(1-c)} + 1$ and $s_1 = 0$. Moreover, denoting $C = 20 + \frac{1}{\sqrt{2}}$, assume that the noise satisfies

$$\left\| \frac{w_1(\cdot)}{\bar{g}_1(\cdot)} \right\|_\infty \leq \varepsilon_1 \frac{u_{\min}}{10C\sqrt{K}} \left(1 + 48 \frac{u_{\max}}{u_{\min}} \right)^{-1}. \quad (4.11)$$

Then, there exists a permutation $\phi_1 : [K] \rightarrow [K]$ such that for $j = 1, \dots, K$, we have

$$\begin{aligned} d_w(\hat{t}_{1,\phi_1(j)}, t_{1,j}) &\leq \varepsilon_1, \\ |\hat{u}_{1,\phi_1(j)} - u_{1,j}| &< \left(4\pi K u_{\max} \left(\frac{2}{\Delta_1(1-c)} + 1 \right) + \frac{u_{\min}}{C\sqrt{K}} \left(1 + 16 \frac{u_{\max}^2}{u_{\min}^2} \right)^{-1} \right) \varepsilon. \end{aligned}$$

Assuming $u_{\min}, u_{\max} = \Theta(1)$, we see from (4.10) that (4.11) is satisfied if $\|w_1\|_\infty \lesssim \mu_1 e^{-\mu_1^2/\Delta_1^2} \frac{\varepsilon_1}{\sqrt{K}}$.

5 Experiments

In this section, we present some numerical experiments for our method pertaining to the error in the recovery of the locations of the spikes. Our setup is as follows. We fix $L = 4$ groups, and consider $K \in \{2, 3, 4, 5\}$. We fix the variance parameters of the Gaussian kernel to be $\mu_L = 0.01$ and $\mu_l = \mu_{l+1}/2$ for $l = 1, \dots, L-1$. The minimum separation parameter Δ is set to 0.05 and m_l is fixed to $1/\Delta + 5$ for each $l = 1, \dots, L$. Furthermore, we fix the sampling parameters as $s_L = 0$, $\varepsilon_L = 0.01$, and for each $l = 1, \dots, L-1$ choose

$$\varepsilon_l = \varepsilon_{l+1}^2, \quad s_l = m_l + \frac{C}{\sqrt{2\pi^2(\mu_{l+1}^2 - \mu_l^2)}} \log^{1/2} \left(\frac{\mu_L}{\mu_l \varepsilon_l} \right) \quad (5.1)$$

for a suitably chosen constant $C > 0$. This is in line with our theory since ε_l is smaller than ε_{l+1} and s_l too is of the form specified in Theorem 5.

In each Monte Carlo run, we choose $K \in \{2, 3, 4, 5\}$, and then randomly generate K spikes for each $l = 1, \dots, L$. In particular, the amplitudes of the spikes are generated by first uniformly sampling values in $[u_{\min}, u_{\max}]$ with $u_{\min} = 3$ and $u_{\max} = 10$, and then randomly assigning each one of them a negative sign with probability $1/2$. Moreover, the spike locations are sampled uniformly at random in $(0, 1)$ with the minimum separation Δ_l for each group ensured to be greater than or equal to Δ . Thereafter, each Fourier sample (for every group) is corrupted with i.i.d Gaussian noise. For each group l , upon obtaining the estimated spike locations $\hat{t}_{l,j}$ and amplitudes $\hat{u}_{l,j}$ for $j = 1, \dots, K$ (via Algorithm 2), we “match” (for each group) the estimated set of spikes with the input spikes based on a simple heuristic. We first find the estimated spike location that has the smallest wrap around distance from an input spike location – this gives us a match. This pair is then removed, and we repeat the process on the remaining sets of spikes. This finally gives us a permutation $\phi : [K] \rightarrow [K]$ where $t_{l,j}$ would ideally be close to $\hat{t}_{l,\phi(j)}$. Finally, we evaluate the performance of our algorithm by examining (a) the maximum wrap around distance $d_{w,l,\max} := \max_j d_w(t_{l,j}, \hat{t}_{l,\phi(j)})$, and (b) the average wrap around distance $d_{w,l,\text{avg}} := (1/K) \sum_{j=1}^K d_w(t_{l,j}, \hat{t}_{l,\phi(j)})$. This is repeated over 400 Monte Carlo trials.

Figures 3, 4 show scatter plots for the above notions of error and the minimum separation Δ_l for each group, in the absence of external noise. Figures 5, 6 show the same, but with external Gaussian noise (standard deviation 5×10^{-5}). Both these plots are for $C = 0.6$ in (5.1), we found this choice to give the best result. We observe that for $l = 1, 2$, the spike locations are recovered near exactly in all the trials and for all choices of K – both in the noiseless and noisy settings. For $l = 3, 4$, the performance is reasonably good for $K = 2, 3$. In particular, for both the noiseless and noisy settings, $d_{w,l,\max}$ is less than 0.05 in at least 82% of the trials while $d_{w,l,\text{avg}}$ is less than 0.05 in at least 93% of trials. While the performance is seen to drop as K increases – especially in terms of $d_{w,l,\max}$ – the performance in terms of $d_{w,l,\text{avg}}$ is still significantly better than $d_{w,l,\max}$. In particular, for $K = 4$, $d_{w,l,\text{avg}}$ is less than 0.05 in at least 86% of the trials (in both noiseless and noisy settings), while for $K = 5$, the same is true in at least 73% of the trials.

As mentioned earlier, we found the choice $C = 0.6$ to give the best performance, in general. In the noiseless scenario however, the choice $C = 1$ results in near exact recovery for all groups as shown in Appendix E (see Figures 7,8). However in the presence of external Gaussian noise of standard deviation 5×10^{-5} , the recovery performance breaks down for groups $l = 2, 3, 4$ (see Figures 9,10). Since we are sampling relatively deeper in the Fourier tail (compared to that when $C = 0.6$), the deconvolution step blows up the noise significantly, leading to the worse recovery performance. Finally, in the noiseless case, we observed that the recovery performance breaks down for larger values of C (i.e., for $C > 4$) due to numerical errors creeping in the deconvolution step.

6 Discussion and Concluding remarks

We now provide a comparison with closely related work and conclude with directions for future research.

6.1 Related work on the multi-kernel unmixing super-resolution problem

Despite its natural role in many practical problems, the study of super-resolution under the presence of multiple kernels has not attracted much theoretical activity until recently. In [31], the authors introduce an interesting variant of the atomic norm approach to sparse measure reconstruction. They prove that the resulting convex optimization problem recovers the original measure in the noiseless case and they provide an interesting error bound in the noisy setting. The assumptions underlying this latter result is a standard separation assumption on the spike localization and a uniform random prior on the Fourier coefficients (when considered to lie in \mathbb{R}/\mathbb{Z}) of the point spread functions. In comparison, our assumptions are quite different. In particular, we do not make any assumption about the randomness of the Fourier coefficients of the point spread function. Moreover, we use Moitra’s Modified MP method instead of the atomic norm penalization considered in [31]. As a main benefit of our approach, we do not need any hyper-parameter tuning when the signal is sufficiently larger than the noise level⁴.

Another interesting work on multi-kernel super-resolution is the technique developed in [42], where the setting is very close to the one of the present paper. A set of relevant modifications of the LASSO estimator and Matching Pursuit method, combined with post-processing techniques, are proposed in [42] and shown to perform well on real datasets. However, to the best of our knowledge, the practical value of these methods is not rigorously supported by theoretical results.

⁴what sufficiently larger means is elaborated on in Theorem 6

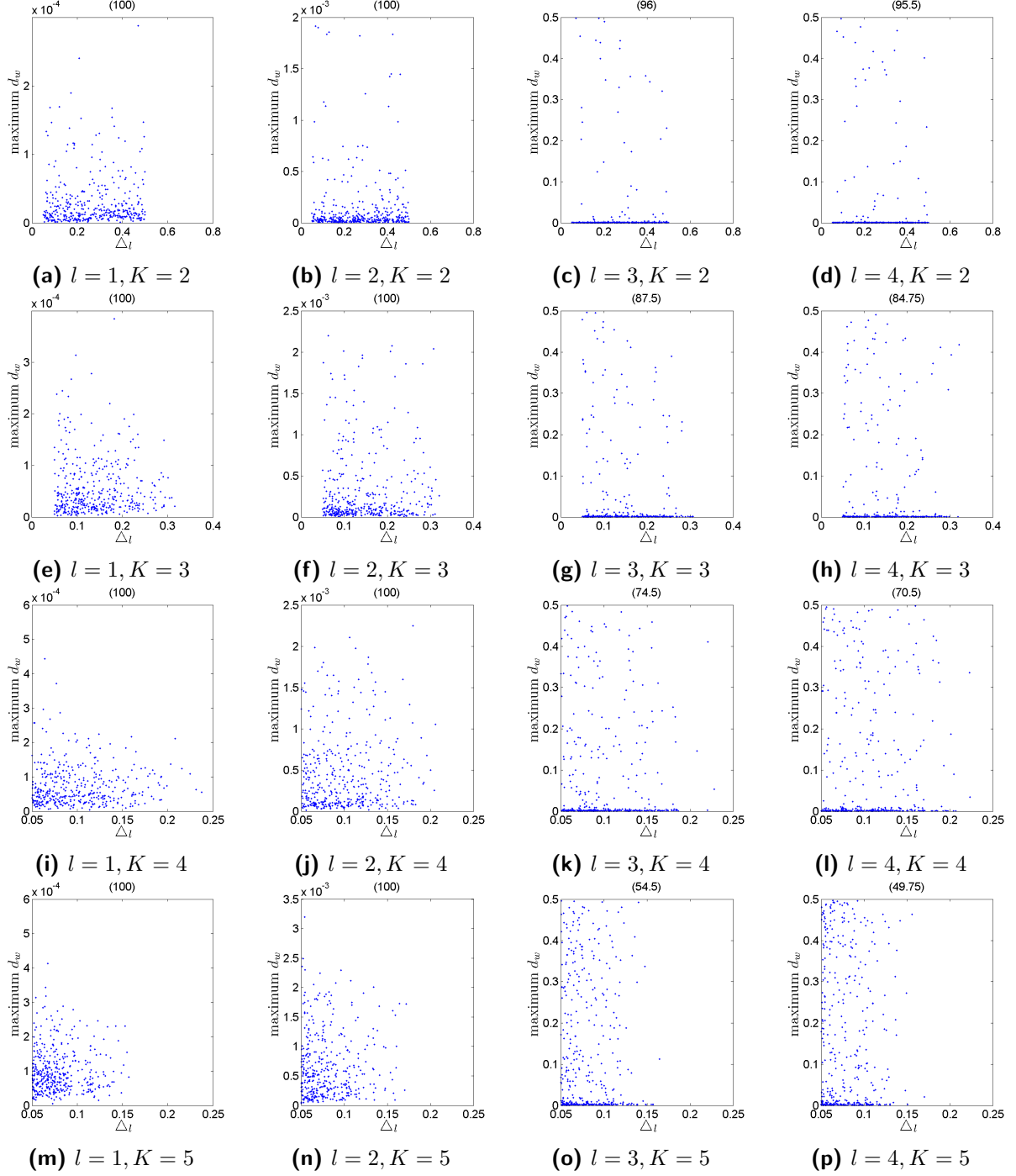


Figure 3: Scatter plots for maximum wrap around error ($d_{w,l,\max}$) v/s minimum separation (Δ_l) for 400 Monte Carlo trials, with no external noise. This is shown for $K \in \{2, 3, 4, 5\}$ with $L = 4$ and $C = 0.6$. For each sub-plot, we mention the percentage of trials with $d_{w,l,\max} \leq 0.05$ in parenthesis.

6.2 Future directions

In this paper, we provide a simple and intuitive algorithm for multi-kernel super-resolution, and also provide strong theoretical results for our approach. There are several directions for extending

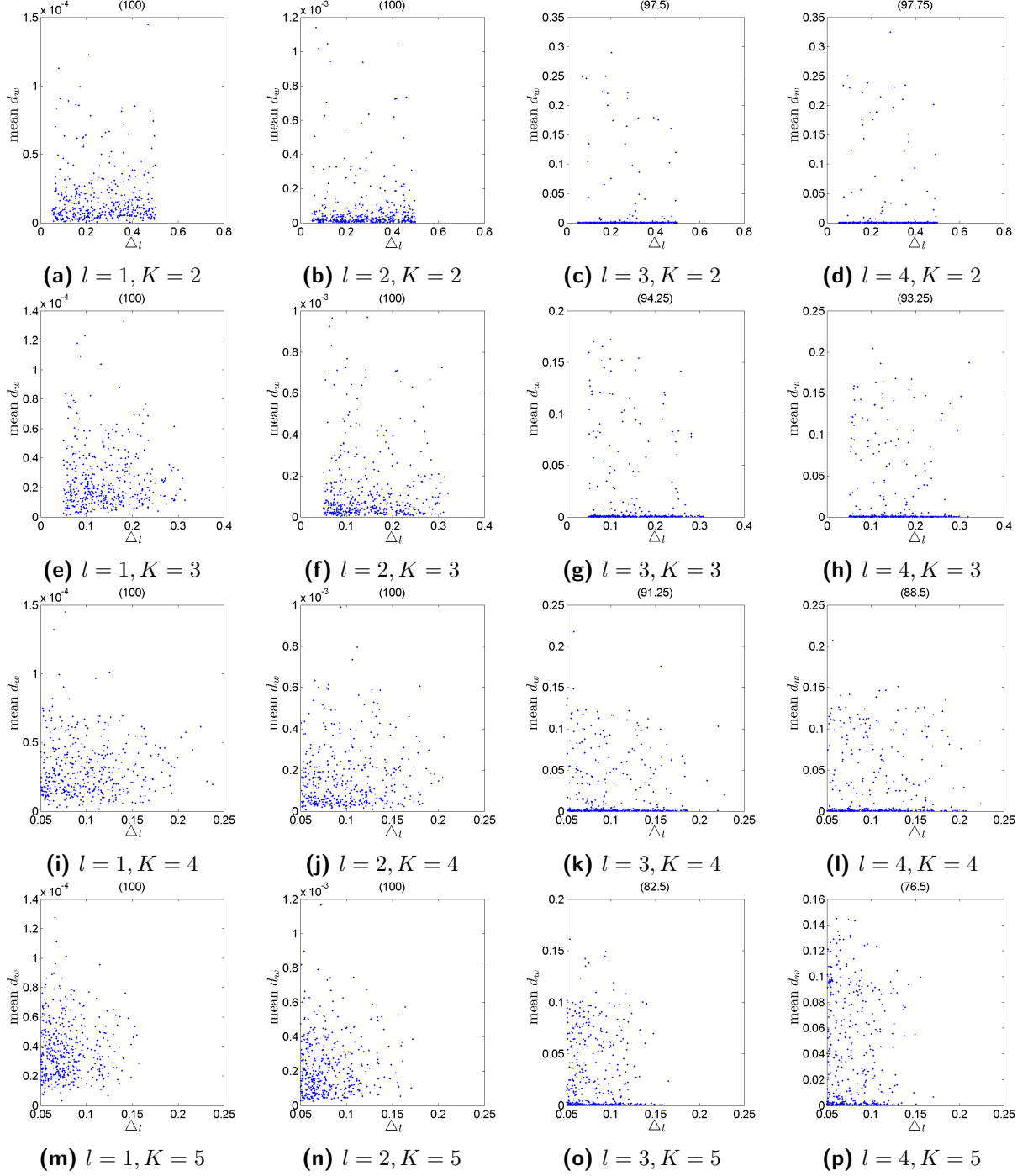


Figure 4: Scatter plots for the mean wrap around error ($d_{w,l,\text{avg}}$) v/s minimum separation (Δ_l) for 400 Monte Carlo trials, with no external noise. This is shown for $K \in \{2, 3, 4, 5\}$ with $L = 4$ and $C = 0.6$. For each sub-plot, we mention the percentage of trials with $d_{w,l,\text{avg}} \leq 0.05$ in parenthesis.

the results in this paper, we list two of them below.

Firstly, our analysis assumes that the kernel variance parameters (i.e., μ_l) are known exactly. In general, the analysis can be extended to the case where upper and lower estimates are available for each μ_l . Hence one could consider estimating the variance terms, and using these estimates

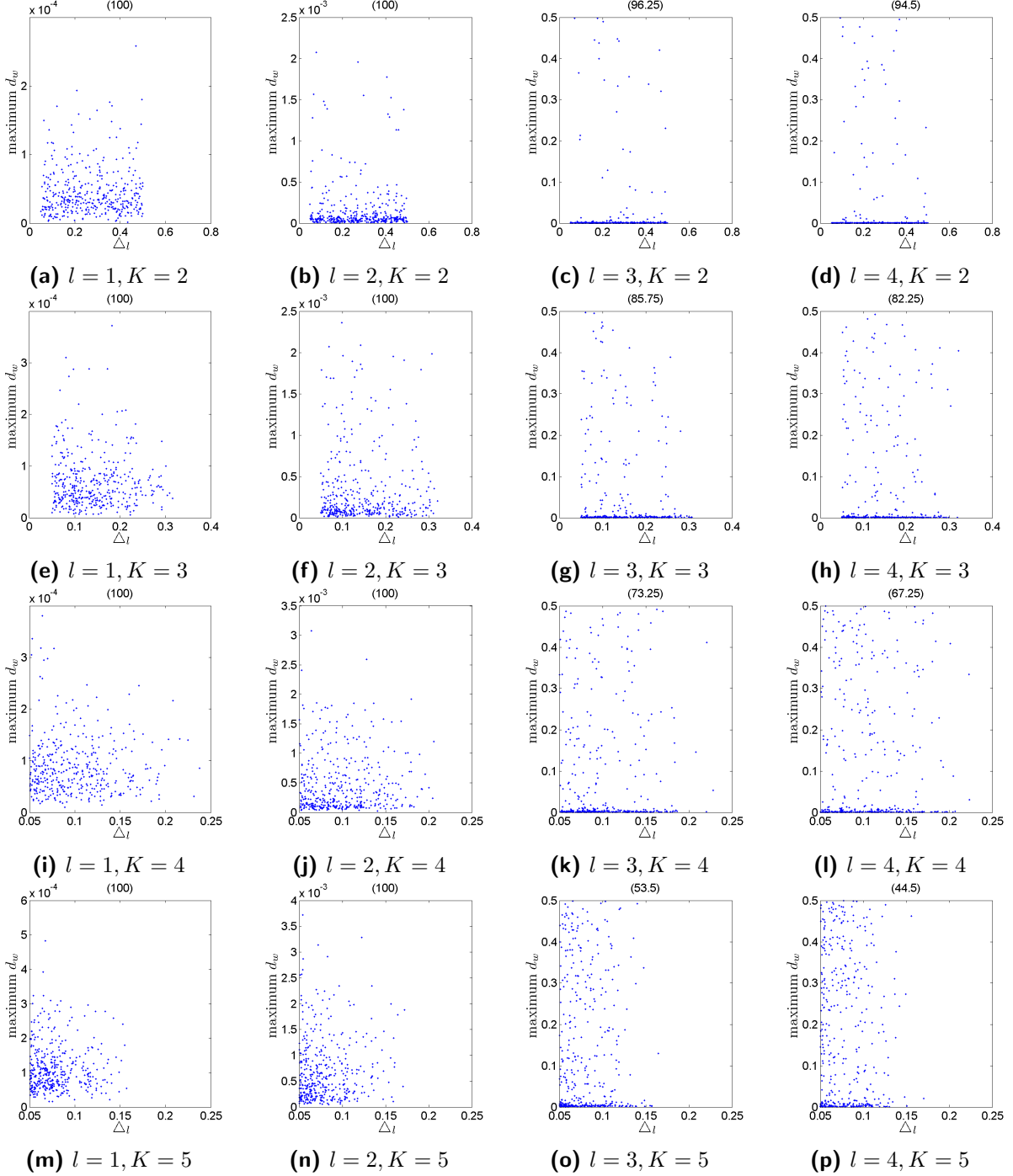


Figure 5: Scatter plots for maximum wrap around error ($d_{w,l,\max}$) v/s minimum separation (Δ_l) for 400 Monte Carlo trials, with external Gaussian noise (standard deviation 5×10^{-5}). This is shown for $K \in \{2, 3, 4, 5\}$ with $L = 4$ and $C = 0.6$. For each sub-plot, we mention the percentage of trials with $d_{w,l,\max} \leq 0.05$ in parenthesis.

with our algorithm. The choice of the method for estimating $(\mu_l)_l$ should be investigated with great care. One possible avenue is to use Lepski's method [20] and its many recent variants and improvements (see for eg., [11]).

Next, it is natural to extend the techniques developed here to the multi-dimensional setting as

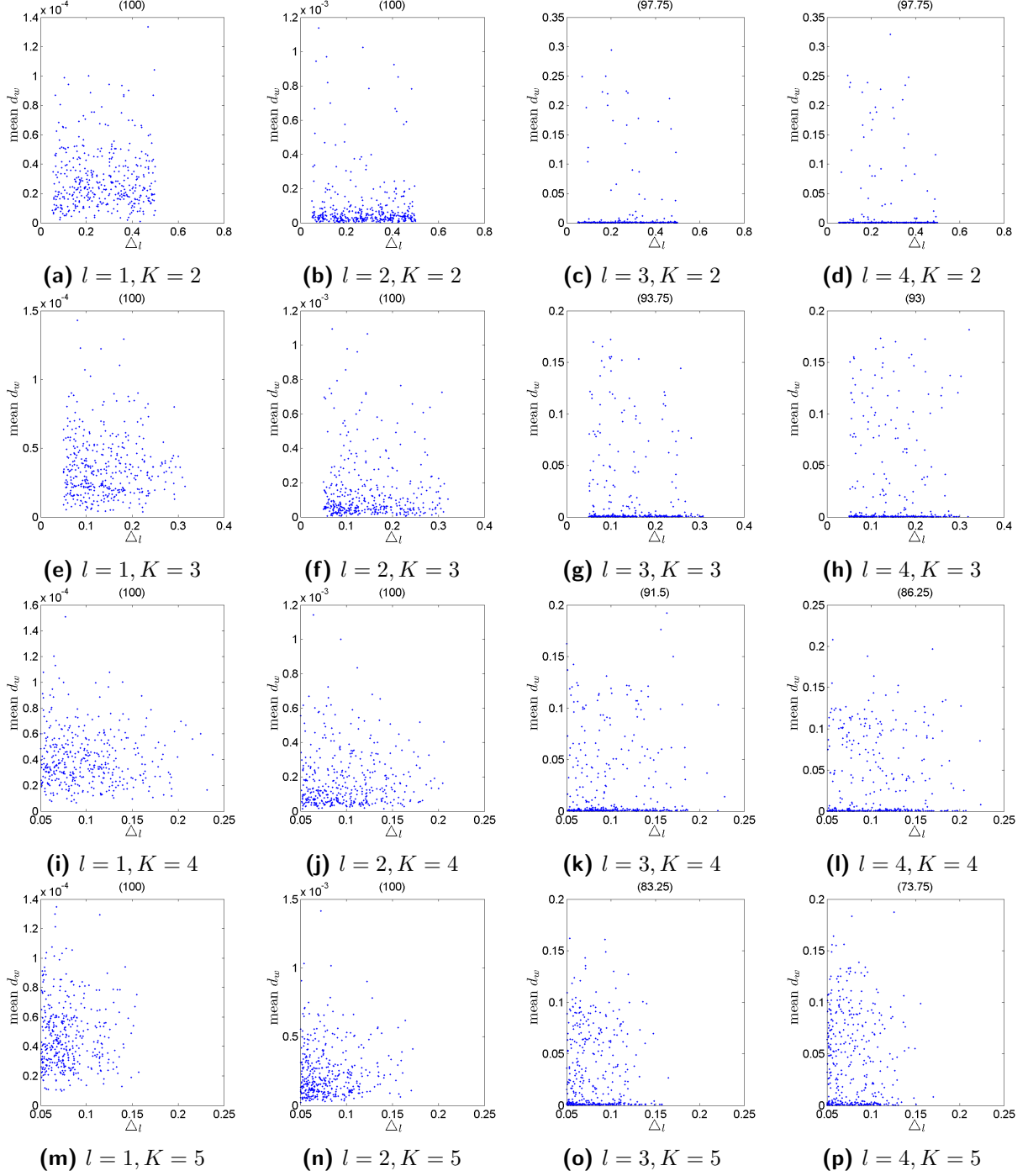


Figure 6: Scatter plots for mean wrap around error ($d_{w,l,\text{avg}}$) v/s minimum separation (Δ_l) for 400 Monte Carlo trials, with external Gaussian noise (standard deviation 5×10^{-5}). This is shown for $K \in \{2, 3, 4, 5\}$ with $L = 4$ and $C = 0.6$. For each sub-plot, we mention the percentage of trials with $d_{w,l,\text{avg}} \leq 0.05$ in parenthesis.

multivariate signals are of high importance in many applications such as DNA sequencing and Mass Spectrometry. This could be investigated using for instance some extension of Prony's method to several dimensions such as in [36], [27] and [22].

References

- [1] Mark D Adams, Chris Fields, and J Craig Venter. *Automated DNA sequencing and analysis*. Elsevier, 2012.
- [2] Jean-Marc Azaïs, Yohann de Castro, and Fabrice Gamboa. Spike detection from inaccurate samplings. *Applied and Computational Harmonic Analysis*, 38(2):177 – 195, 2015.
- [3] Tamir Bendory, Shai Dekel, and Arie Feuer. Robust recovery of stream of pulses using convex optimization. *Journal of mathematical analysis and applications*, 442(2):511–536, 2016.
- [4] Anthony J Berno. A graph theoretic approach to the analysis of dna sequencing data. *Genome Research*, 6(2):80–91, 1996.
- [5] Nicholas Boyd, Geoffrey Schiebinger, and Benjamin Recht. The alternating descent conditional gradient method for sparse inverse problems. *SIAM Journal on Optimization*, 27(2):616–639, 2017.
- [6] Emery N Brown, Robert E Kass, and Partha P Mitra. Multiple neural spike train data analysis: state-of-the-art and future challenges. *Nature neuroscience*, 7(5):456, 2004.
- [7] Pierpaolo Brutti, Christopher Genovese, Christopher J Miller, Robert C Nichol, and Larry Wasserman. Spike hunting in galaxy spectra. Technical report, Libera Università Internazionale degli Studi Sociali Guido Carli di Roma, <http://www.stat.cmu.edu/tr/tr828/tr828.html>, 2005.
- [8] Peter Bühlmann and Sara Van De Geer. *Statistics for high-dimensional data: methods, theory and applications*. Springer Science & Business Media, 2011.
- [9] Emmanuel J Candès and Carlos Fernandez-Granda. Super-resolution from noisy data. *Journal of Fourier Analysis and Applications*, 19(6):1229–1254, 2013.
- [10] Emmanuel J Candès and Carlos Fernandez-Granda. Towards a mathematical theory of super-resolution. *Communications on Pure and Applied Mathematics*, 67(6):906–956, 2014.
- [11] Fabienne Comte and Claire Lacour. Anisotropic adaptive kernel deconvolution. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 49(2):569–609, 2013.
- [12] B. G. R. de Prony. Essay experimental et analytique: sur les lois de la dilatabilité de fluides élastique et sur celles de la force expansive de la vapeur de l’alcool, à différentes températures. *Journal de l’école Polytechnique*, pages 24–76, 1795.
- [13] Quentin Denoyelle, Vincent Duval, and Gabriel Peyré. Asymptotic of sparse support recovery for positive measures. *Journal of Physics: Conference Series*, 657(1):012013, 2015.
- [14] Vincent Duval and Gabriel Peyré. Exact support recovery for sparse spikes deconvolution. *Foundations of Computational Mathematics*, 15(5):1315–1355, 2015.
- [15] Vincent Duval and Gabriel Peyré. Sparse regularization on thin grids i: the lasso. *Inverse Problems*, 33(5):055008, 2017.
- [16] Vincent Duval and Gabriel Peyré. Sparse spikes super-resolution on thin grids ii: the continuous basis pursuit. *Inverse Problems*, 33(9):095008, 2017.

- [17] Yonina C Eldar and Gitta Kutyniok. *Compressed sensing: theory and applications*. Cambridge University Press, 2012.
- [18] C. Fernandez-Granda. Support detection in super-resolution. *International Conference on Sampling Theory and Appl.*, pages 145–148, 2013.
- [19] Simon Foucart and Holger Rauhut. *A Mathematical Introduction to Compressive Sensing*. Birkhäuser Basel, 2013.
- [20] Alexander Goldenshluger and Oleg Lepski. Bandwidth selection in kernel density estimation: oracle inequalities and adaptive minimax optimality. *The Annals of Statistics*, 39(3):1608–1632, 2011.
- [21] Hayit Greenspan. Super-resolution in medical imaging. *The Computer Journal*, 52(1):43–63, 2009.
- [22] Jouhayna Harmouch, Houssam Khalil, and Bernard Mourrain. Structured low rank decomposition of multivariate hankel matrices. *Linear Algebra and its Applications*, 2017.
- [23] R. Hettich and K. O. Kortanek. Semi-infinite programming: Theory, methods, and applications. *SIAM Review*, 35(3):380–429, 1993.
- [24] Y. Hua and T. K. Sarkar. Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 38(5):814–824, 1990.
- [25] Jiaqing Huang, Mingzhai Sun, Kristyn Gumpfer, Yuejie Chi, and Jianjie Ma. 3d multifocus astigmatism and compressed sensing (3d macs) based superresolution reconstruction. *Biomedical optics express*, 6(3):902–917, 2015.
- [26] V. Khaidukov, E. Landa, and T. J. Moser. Diffraction imaging by focusing/defocusing: An outlook on seismic superresolution. *GEOPHYSICS*, 69(6):1478–1490, 2004.
- [27] Stefan Kunis, Thomas Peter, Tim Römer, and Ulrich von der Ohe. A multivariate generalization of prony’s method. *Linear Algebra and its Applications*, 490:31–47, 2016.
- [28] Michael S Lewicki. A review of methods for spike sorting: the detection and classification of neural action potentials. *Network: Computation in Neural Systems*, 9(4):R53–R78, 1998.
- [29] Lei Li. Dna sequencing and parametric deconvolution. *Statistica Sinica*, pages 179–202, 2002.
- [30] Lei M Li and Terence P Speed. Deconvolution of sparse positive spikes. *Journal of Computational and Graphical Statistics*, 13(4):853–870, 2004.
- [31] Yuanxin Li and Yuejie Chi. Stable separation and super-resolution of mixture models. *Applied and Computational Harmonic Analysis*, 2017.
- [32] Han Liu, Advisors: Larry Wasserman, Christopher Genovese, and John Lafferty. Estimating galaxy spectral functions using sparse composite models, 2007.
- [33] Jie Lu. Photopik detection and quantification using wavelet analysis. Master’s thesis, University of Illinois at Urbana-Champaign, 2013.

- [34] Ivan Markovsky. Structured low-rank approximation and its applications. *Automatica*, 44(4):891–909, 2008.
- [35] A. Moitra. Super-resolution, extremal functions and the condition number of vandermonde matrices. In *Proceedings of the Forty-seventh Annual ACM Symposium on Theory of Computing*, STOC ’15, pages 821–830, 2015.
- [36] Thomas Peter, Gerlind Plonka, and Robert Schaback. Prony’s method for multivariate signals. *PAMM*, 15(1):665–666, 2015.
- [37] Gerlind Plonka and Manfred Tasche. Prony methods for recovery of structured functions. *GAMM-Mitteilungen*, 37(2):239–258, 2014.
- [38] Daniel Potts and Manfred Tasche. Parameter estimation for exponential sums by approximate prony method. *Signal Processing*, 90(5):1631 – 1642, 2010. Special Section on Statistical Signal & Array Processing.
- [39] Daniel Potts and Manfred Tasche. Parameter estimation for nonincreasing exponential sums by prony-like methods. *Linear Algebra and its Applications*, 439(4):1024 – 1039, 2013.
- [40] K.G. Puschmann and F. Kneer. On super-resolution in astronomical imaging. *A&A*, 436(1):373–378, 2005.
- [41] Geoffrey Schiebinger, Elina Robeva, and Benjamin Recht. Superresolution without separation. *Information and Inference: A Journal of the IMA*, 7(1), 2018.
- [42] Martin Slawski and Matthias Hein. Sparse recovery for protein mass spectrometry data. In *Practical Applications of Sparse Modeling.*, pages 79–98. MIT Press, 2014.
- [43] SnapGene. View and edit DNA sequence traces. http://www.snapgene.com/products/snapgene_viewer/, 2018.
- [44] GW Stewart and JG Sun. *Matrix Perturbation Theory*. Academic Press Inc., 1990.
- [45] Jacob Stinnett. Bayesian algorithms for automated isotope identification. Master’s thesis, University of Illinois at Urbana-Champaign, 2014.
- [46] Jacob Benjamin Stinnett. *Automated isotope identification algorithms for low-resolution gamma spectrometers*. PhD thesis, University of Illinois at Urbana-Champaign, 2016.
- [47] G. Tang, B. N. Bhaskar, and B. Recht. Near minimax line spectral estimation. *IEEE Transactions on Information Theory*, 61(1):499–512, 2015.
- [48] Gongguo Tang, Badri Narayan Bhaskar, Parikshit Shah, and Benjamin Recht. Compressed sensing off the grid. *IEEE transactions on information theory*, 59(11):7465–7490, 2013.
- [49] Holger Wendland. *Scattered Data Approximation*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, 2004.

A Modified matrix pencil method

A.1 Proof of Moitra's theorem

We now outline the steps of Moitra's proof [35, Theorem 2.8] for completeness. In particular, we note that the result in [35, Theorem 2.8] does not detail the constants appearing in the bounds, while we will do so here.

To begin with, recall the notion of chordal metric for measuring distance between complex numbers.

Definition 1. *The chordal metric for $u, v \in \mathbb{C}$ is defined as*

$$\chi(u, v) := \frac{|u - v|}{\sqrt{1 + |u|^2} \sqrt{1 + |v|^2}}.$$

Denoting $s(u), s(v) \in \mathbb{S}^2$ to be points with u, v as their respective stereographic projections on the plane, we also have $\chi(u, v) = \frac{1}{2} \|s(u) - s(v)\|$. It is useful to note that if $(u, v) \in \mathbb{R}^2$ is the stereographic projection of a point $(a, b, c) \in \mathbb{S}^2$, then

$$(a, b, c) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right).$$

Definition 2. *Let $(\lambda_i)_{i=1}^n$ denote generalized eigenvalues of (A, B) , and also let $(\hat{\lambda}_i)_{i=1}^n$ denote generalized eigenvalues of (\hat{A}, \hat{B}) . Then the matching distance with respect to χ is defined as*

$$md_\chi[(A, B), (\hat{A}, \hat{B})] := \min_{\phi} \max_i \chi(\lambda_i, \hat{\lambda}_{\phi(i)}),$$

where $\phi : [n] \rightarrow [n]$ denotes a permutation.

The key to the analysis in [35, Theorem 2.8] are the following two technical results.

Theorem 7. ([44]) *Let (A, B) and (\tilde{A}, \tilde{B}) be regular pairs and further suppose that for some nonsingular X we have $(XAX^H, XBX^H) = (I, D)$ where D is diagonal. Also let e_i be the i th row of $X(A - \hat{A})X^H$ and let f_i be the i th row of $X(B - \hat{B})X^H$ and set $\rho = \max_i \{\|e_i\|_1 + \|f_i\|_1\}$. If the following regions*

$$\mathcal{G}_i = \{\mu \mid \chi(\alpha_i, \mu) \leq \rho\}$$

are disjoint then the matching distance of the generalized eigenvalues of (\hat{A}, \hat{B}) to $\{D_{ii}\}_i$ is at most ρ with respect to the chordal metric.

Theorem 8. ([35, Theorem 1.1]) *Provided that $m > \frac{1}{\Delta} + 1$, we have $\sigma_{max}^2 \leq m + \frac{1}{\Delta} - 1$ and $\sigma_{min}^2 \geq m - \frac{1}{\Delta} - 1$. Consequently, the condition number of V satisfies*

$$\kappa^2 \leq \frac{m + \frac{1}{\Delta} - 1}{m - \frac{1}{\Delta} - 1}.$$

Step 1: Recovering each t_j . Let \hat{U} denote the top left singular vectors of \tilde{H}_0 . The following lemma shows that there exists an orthonormal basis for the column span of V that is well aligned to \hat{U} .

Lemma 1. ([35, Lemma 2.7]) *There exists a matrix U such that*

$$\|U - \hat{U}\|_2 \leq \frac{2\|E\|_2}{\sigma_K(VD_uV^H) - \|E\|_2}$$

and the columns of U form an orthonormal basis for V and those of \hat{U} form an orthogonal basis of the K “largest” singular vectors of \tilde{H}_0 .

Remark 1. *The original statement in [35, Lemma 2.7] has minor inaccuracies in our opinion, namely that the denominator term should be $\sigma_K(VD_uV^H + E)$ and not $\sigma_K(VD_uV^H)$ as this is what we obtain from Wedin’s bound. Since $\sigma_K(VD_uV^H + E) \geq \sigma_K(VD_uV^H) - \|E\|_2$ (Weyl’s inequality), the statement in Lemma 1 follows.*

By projecting H_0, H_1 to U we obtain

$$A = U^H H_0 U = U^H V D_u V^H U \quad \text{and} \quad B = U^H H_1 U = U^H V D_u D_\alpha V^H U.$$

As easily checked, the generalized eigenvalues of (B, A) are $\alpha_1, \dots, \alpha_K$. Similarly,

$$\hat{A} = \hat{U}^H \tilde{H}_0 \hat{U} = \hat{U}^H (V D_u V^H + E) \hat{U} \quad \text{and} \quad \hat{B} = \hat{U}^H \tilde{H}_1 \hat{U} = \hat{U}^H (V D_u D_\alpha V^H + F) \hat{U}.$$

The generalized eigenvalues of (\hat{B}, \hat{A}) are perturbed versions of that of A and B . Since $\sigma_K(VD_uV^H) \geq \sigma_{\min}^2 u_{\min}$, therefore Lemma 1 gives us the bound

$$\|U - \hat{U}\|_2 \leq \frac{2\|E\|_2}{\sigma_{\min}^2 u_{\min} - \|E\|_2} = \tau_E.$$

We will denote τ_F to be equal to the same bound as above, with E replaced by F . Let $X = D_u^{-\frac{1}{2}} V^\dagger U$, clearly X is non-singular and $XAX^H = I$ and $XBX^H = D_\alpha$. We now bound the ℓ_1 -norm of each row of $X(A - \hat{A})X^H$ and $X(B - \hat{B})X^H$ using what we have seen so far. To begin with, following the steps in [35], we obtain

$$\|X(A - \hat{A})X^H\|_2 \leq \frac{\|E\|_2 + \sigma_{\max}^2 \tau_E (2 + \tau_E)}{u_{\min} \sigma_{\min}^2} = \frac{\|E\|_2}{u_{\min} \sigma_{\min}^2} + \kappa^2 \frac{u_{\max}}{u_{\min}} \tau_E (2 + \tau_E).$$

The same bound holds on $X(B - \hat{B})X^H$ with E replaced by F . Let e_i, f_i denote the i^{th} row of $X(A - \hat{A})X^H, X(B - \hat{B})X^H$ respectively and let $\rho = \max_i \{\|e_i\|_1 + \|f_i\|_1\}$. Using the fact $\rho \leq \sqrt{K} \max_i \{\|e_i\|_2 + \|f_i\|_2\}$ we get

$$\rho \leq \sqrt{K} \left(\frac{\|E\|_2 + \|F\|_2}{u_{\min} \sigma_{\min}^2} + \frac{\kappa^2 u_{\max}}{u_{\min}} [\tau_E (2 + \tau_E) + \tau_F (2 + \tau_F)] \right). \quad (\text{A.1})$$

Assume that $\|E\|_2, \|F\|_2 \leq \delta \frac{u_{\min} \sigma_{\min}^2}{2}$ for some $\delta \in (0, 1)$. Then,

$$\tau_E (2 + \tau_E) = \frac{\delta}{1 - \delta/2} \left(2 + \frac{\delta}{1 - \delta/2} \right) = \frac{2\delta}{(1 - \delta/2)^2} \leq 8\delta.$$

Similarly, $\tau_F(2 + \tau_F) \leq 8\delta$. Therefore we have

$$\rho \leq \sqrt{K} \delta \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}} \right).$$

On the other hand, one can easily verify that $\|E\|_2, \|F\|_2 \leq m\eta_{\max}$. Therefore, if $\eta_{\max} \leq \delta \frac{u_{\min} \sigma_{\min}^2}{2m}$, then we get $\rho \leq \sqrt{K} \delta \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}} \right)$.

Now let us note that $\chi(\alpha_i, \alpha_j) \geq 2d_w(t_i, t_j) \geq 2\Delta$ (see Appendix B.1) for $i \neq j$. So if u is such that $\chi(\alpha_i, u) \leq \epsilon$, then this would imply $\chi(\alpha_j, u) \geq 2\Delta - \epsilon$, and hence if additionally $\epsilon < \Delta$ holds, then the regions $\{u \mid \chi(\alpha_i, u) \leq \epsilon\}$ will be disjoint. Denote $(\hat{\lambda}_j)_{j=1}^n$ to be the generalized eigenvalues of (\hat{A}, \hat{B}) . Therefore, if $\delta \in (0, 1)$ satisfies $\delta < \frac{\Delta}{\sqrt{K}} \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}} \right)^{-1}$ (note that the bound is already less than 1), then from Theorem 7 we know that there exists a permutation $\phi : [k] \rightarrow [k]$ such that $\chi(\hat{\lambda}_{\phi(i)}, \alpha_i) \leq \sqrt{K} \delta \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}} \right)$ for $i = 1, \dots, n$. Let $\hat{\alpha}_i = \hat{\lambda}_i / |\hat{\lambda}_i|$, and set $\sqrt{K} \delta \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}} \right) = \varepsilon/C$ where $C = 10 + \frac{1}{2\sqrt{2}}$ and $0 \leq \varepsilon < \min\{1, C\Delta\}$. This means $\chi(\hat{\lambda}_{\phi(i)}, \alpha_i) \leq \varepsilon/C$ holds, and so, from⁵ Proposition 1 in Appendix B.2, this implies $d_w(\hat{t}_{\phi(i)}, t_i) \leq \varepsilon$. Putting it together with the earlier condition on δ , we get that if

$$\eta_{\max} \leq \varepsilon \frac{u_{\min} \sigma_{\min}^2}{2mC\sqrt{K}} \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}} \right)^{-1} \quad (\text{A.2})$$

is satisfied, for $0 \leq \varepsilon < \min\{1, C\Delta\}$, then $d_w(\hat{t}_{\phi(i)}, t_i) \leq \varepsilon$.

Step 2: Recovering each u'_j . Note that $d_w(\hat{t}_i, \hat{t}_j) \geq \Delta - 2\varepsilon$ for all $i \neq j$, so we assume $\varepsilon < \Delta/2$ from now. Also recall that we form the Vandermonde matrix $\hat{V} \in \mathbb{C}^{m \times K}$ using $\hat{\alpha}_j = \exp(-i2\pi \hat{t}_j)$. Then, the estimate $\hat{u} = \hat{V}^\dagger v \in \mathbb{C}^K$ satisfies

$$\hat{u}' = \hat{V}^\dagger v = \hat{V}^\dagger V u' + \hat{V}^\dagger \eta.$$

Denote $\tilde{\phi} = \phi^{-1}$ and let $u'_{\tilde{\phi}} \in \mathbb{C}^K$ denote the permuted version of u' w.r.t $\tilde{\phi}$. Also, let $V_{\tilde{\phi}}$ be formed by permuting the columns of V w.r.t $\tilde{\phi}$. Then,

$$\begin{aligned} \hat{u}' - u'_{\tilde{\phi}} &= (\hat{V}^\dagger V_{\tilde{\phi}} u'_{\tilde{\phi}} - u'_{\tilde{\phi}}) + \hat{V}^\dagger \eta \\ &= (\hat{V}^\dagger V_{\tilde{\phi}} - \hat{V}^\dagger \hat{V}) u'_{\tilde{\phi}} + \hat{V}^\dagger \eta \\ \Rightarrow \|\hat{u}' - u'_{\tilde{\phi}}\|_2 &\leq \|\hat{V}^\dagger\|_2 (\|V_{\tilde{\phi}} - \hat{V}\|_2 \|u'_{\tilde{\phi}}\|_2 + \|\eta\|_2). \end{aligned} \quad (\text{A.3})$$

Now, $\hat{V}^\dagger = (\hat{V}^H \hat{V})^{-1} \hat{V}^H$ so $\|\hat{V}^\dagger\|_2 = (\sigma_{\min}(\hat{V}))^{-1} \leq (m - \frac{1}{\Delta - 2\varepsilon} - 1)^{-1/2}$ (cf., Theorem 8). Next, the magnitude of each entry in $V_{\tilde{\phi}} - \hat{V}$ can be verified to be upper bounded by $2\pi m \max_i d_w(t_i, \hat{t}_{\tilde{\phi}(i)}) \leq 2\pi m \varepsilon$ (see Proposition 2 in Appendix B.3). Thus, $\|V_{\tilde{\phi}} - \hat{V}\|_2 \leq \|V_{\tilde{\phi}} - \hat{V}\|_F \leq 2\pi m^{3/2} K^{1/2} \varepsilon$. Also, $\|u'_{\tilde{\phi}}\|_2 \leq \sqrt{K} u_{\max}$ and using (A.2), we obtain

$$\|\eta\|_2 \leq \sqrt{m} \eta_{\max} \leq \varepsilon \frac{u_{\min} \sigma_{\min}^2}{2C\sqrt{mK}} \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}} \right)^{-1}.$$

⁵The Proposition requires $0 \leq \varepsilon/C \leq 1/4$, which is the case here.

Plugging these bounds in (A.3), we obtain

$$\|\hat{u}' - u'_{\tilde{\phi}}\|_{\infty} \leq \|\hat{u}' - u'_{\tilde{\phi}}\|_2 \leq \frac{2\pi m^{3/2} K u_{\max} \varepsilon + \varepsilon \frac{u_{\min} \sigma_{\min}^2}{2C\sqrt{mK}} \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}}\right)^{-1}}{\left(m - \frac{1}{\Delta - 2\varepsilon} - 1\right)^{1/2}}. \quad (\text{A.4})$$

Step 3: Recovering each u_j . Given $\hat{u}' \in \mathbb{C}^K$, we obtain our final estimate $\hat{u}_j = \exp(-\iota 2\pi s_0 \hat{t}_j) \hat{u}'_j$ for $j = 1, \dots, K$. Denoting

$$\begin{aligned} \hat{D} &= \text{diag}(\exp(-\iota 2\pi s_0 \hat{t}_1), \dots, \exp(-\iota 2\pi s_0 \hat{t}_K)), \\ D_{\tilde{\phi}} &= \text{diag}(\exp(-\iota 2\pi s_0 t_{\tilde{\phi}(1)}), \dots, \exp(-\iota 2\pi s_0 t_{\tilde{\phi}(K)})), \end{aligned}$$

note that $\hat{u} = \hat{D} \hat{u}'$ and $u_{\tilde{\phi}} = D_{\tilde{\phi}} u'_{\tilde{\phi}}$. From this, we obtain

$$\begin{aligned} \hat{u} - u_{\tilde{\phi}} &= \hat{D} \hat{u}' - D_{\tilde{\phi}} u'_{\tilde{\phi}} \\ &= \hat{D} \hat{u}' - \hat{D} u'_{\tilde{\phi}} + \hat{D} u'_{\tilde{\phi}} - D_{\tilde{\phi}} u'_{\tilde{\phi}} \\ &= \hat{D} (\hat{u}' - u'_{\tilde{\phi}}) + (\hat{D} - D_{\tilde{\phi}}) u'_{\tilde{\phi}} \\ \Rightarrow \|\hat{u} - u_{\tilde{\phi}}\|_{\infty} &\leq \|\hat{u}' - u'_{\tilde{\phi}}\|_{\infty} + \|\hat{D} - D_{\tilde{\phi}}\|_{\infty} \|u'_{\tilde{\phi}}\|_{\infty} \end{aligned} \quad (\text{A.5})$$

where in the last line, we used $\|\hat{D}\|_{\infty} = 1$. Using Proposition 2 in Appendix B.3, we readily obtain the bound $\|\hat{D} - D_{\tilde{\phi}}\|_{\infty} \leq 2\pi s_0 \max_i d_w(\hat{t}_i, t_{\tilde{\phi}(i)}) \leq 2\pi s_0 \varepsilon$. Using $\|u'_{\tilde{\phi}}\|_{\infty} = u_{\max}$ and (A.4) in (A.5), we finally obtain

$$\|\hat{u} - u_{\tilde{\phi}}\|_{\infty} \leq \frac{2\pi m^{3/2} K u_{\max} \varepsilon + \varepsilon \frac{u_{\min} \sigma_{\min}^2}{2C\sqrt{mK}} \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}}\right)^{-1}}{\left(m - \frac{1}{\Delta - 2\varepsilon} - 1\right)^{1/2}} + 2\pi u_{\max} s_0 \varepsilon. \quad (\text{A.6})$$

A.2 Proof of Corollary 1

We need only make the following simple observations. Firstly, $\sigma_{\min}^2 \leq \sigma_{\max}^2 \leq m + \frac{1}{\Delta} + 1 < 2m$, since $m > 1/\Delta + 1$. Also, since $\kappa^2 \geq 1$, we get

$$\frac{u_{\min} \sigma_{\min}^2 \varepsilon}{2C\sqrt{mK}} \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}}\right)^{-1} < \frac{u_{\min}}{C\sqrt{K}} \left(1 + 16 \frac{u_{\max}}{u_{\min}}\right)^{-1} \sqrt{m} \varepsilon.$$

Next, one can easily verify that for $m \geq \frac{2}{\Delta - 2\varepsilon} + 1$, we have $\frac{\sqrt{m}}{\sqrt{m - \frac{1}{\Delta - 2\varepsilon} - 1}} \leq 2$. Using these observations in (2.7), along with $m < \frac{2}{\Delta(1-c)} + 1$, we obtain the stated bound on $\|\hat{u}_{\phi} - u\|_{\infty}$.

Next, let us note that the condition $m \geq \frac{2}{\Delta - 2\varepsilon} + 1 \geq \frac{2}{\Delta} + 1$ leads to the following simple observations.

(i)

$$\frac{m}{\sigma_{\min}^2} \leq \frac{m}{m - \frac{1}{\Delta} - 1} \leq \frac{1}{1 - \frac{1/\Delta + 1}{2/\Delta + 1}} \leq 5/2.$$

(ii)

$$\kappa^2 = \frac{m + \frac{1}{\Delta} - 1}{m - \frac{1}{\Delta} - 1} \leq 3.$$

Plugging these in (2.6), we have that it suffices for η_{\max} to satisfy

$$\eta_{\max} \leq \varepsilon \frac{u_{\min}}{5C\sqrt{K}} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right)^{-1} \leq \varepsilon \frac{u_{\min}\sigma_{\min}^2}{2mC\sqrt{K}} \left(1 + 16 \kappa^2 \frac{u_{\max}}{u_{\min}}\right)^{-1}.$$

B Auxiliary results

B.1 Useful relation involving wrap around metric and chordal metric for points on the unit disk

For $t_1, t_2 \in [0, 1)$ let $\alpha_1 = \exp(i2\pi t_1)$, $\alpha_2 = \exp(i2\pi t_2)$ denote their representations on the unit disk. We have

$$\begin{aligned} |\alpha_1 - \alpha_2| &= |1 - \exp(i2\pi(t_2 - t_1))| \\ &= \sqrt{(1 - \cos(2\pi(t_2 - t_1)))^2 + \sin^2(2\pi(t_2 - t_1))} \\ &= \sqrt{2 - 2\cos(2\pi(t_2 - t_1))} \\ &= 2|\sin(\pi(t_2 - t_1))| \\ &= 2\sin(\pi|t_2 - t_1|) \quad (\text{Since } t_2 - t_1 \in (-1, 1)) \\ &= 2\sin(\pi - \pi|t_2 - t_1|) = 2\sin(\pi d_w(t_1, t_2)). \end{aligned}$$

Since $\sin x \leq x$ for $x \geq 0$, therefore we obtain $|\alpha_1 - \alpha_2| \leq 2\pi d_w(t_1, t_2)$. Also, since $d_w(t_1, t_2) \in [0, 1/2]$ and $\sin x \geq \frac{2x}{\pi}$ for $x \in [0, \pi/2]$, we obtain $d_w(t_1, t_2) \leq |\alpha_1 - \alpha_2|/4$. To summarize,

$$\frac{|\alpha_1 - \alpha_2|}{2\pi} \leq d_w(t_1, t_2) \leq \frac{|\alpha_1 - \alpha_2|}{4}. \quad (\text{B.1})$$

Finally, from the definition of chordal metric (see Definition 1) we know that $\chi(\alpha_1, \alpha_2) = |\alpha_1 - \alpha_2|/2$. Thus from (B.1), we obtain

$$\frac{\chi(\alpha_1, \alpha_2)}{\pi} \leq d_w(t_1, t_2) \leq \frac{\chi(\alpha_1, \alpha_2)}{2}. \quad (\text{B.2})$$

B.2 Useful relation involving wrap around metric and chordal metric when one point lies on the unit disk, and the other does not

We will prove the following useful Proposition.

Proposition 1. *For $\alpha_1, \alpha_2 \in \mathbb{C}$ where $|\alpha_1| = 1$, let $t_1, t_2 \in [0, 1)$ be such that $\alpha_1 = \exp(i2\pi t_1)$, $\alpha_2/|\alpha_2| = \exp(i2\pi t_2)$. If $\chi(\alpha_1, \alpha_2) \leq \epsilon$ for some $0 \leq \epsilon \leq 1/4$, then $d_w(t_1, t_2) \leq (20 + \frac{1}{\sqrt{2}})\epsilon$.*

Proof. We begin by representing α_1, α_2 in Cartesian coordinates in \mathbb{R}^2 where

$$\alpha_1 = (a, b), \quad \alpha_2 = (x, y).$$

Let $s(\alpha_1), s(\alpha_2) \in \mathbb{S}^2$ be such that α_1, α_2 are their respective stereographic projections. Then,

$$\begin{aligned} s(\alpha_2) &= \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right) \\ s(\alpha_1) &= (a, b, 0). \end{aligned}$$

Since $\chi(\alpha_1, \alpha_2) = \frac{\|s(\alpha_1) - s(\alpha_2)\|_2}{2} \leq \epsilon$, this implies

$$\frac{|-1 + x^2 + y^2|}{1 + x^2 + y^2} \leq 2\epsilon, \quad \left| \frac{2x}{1 + x^2 + y^2} - a \right| \leq 2\epsilon, \quad \left| \frac{2y}{1 + x^2 + y^2} - b \right| \leq 2\epsilon. \quad (\text{B.3})$$

Let us assume $\epsilon < 1/2$ from now.

- The first inequality in (B.3) implies

$$\frac{1 - 2\epsilon}{1 + 2\epsilon} \leq x^2 + y^2 \leq \frac{1 + 2\epsilon}{1 - 2\epsilon}. \quad (\text{B.4})$$

- The second inequality in (B.3) implies

$$(a - 2\epsilon) \left(\frac{1 + x^2 + y^2}{2} \right) \leq x \leq (a + 2\epsilon) \left(\frac{1 + x^2 + y^2}{2} \right). \quad (\text{B.5})$$

- The third inequality in (B.3) implies

$$(b - 2\epsilon) \left(\frac{1 + x^2 + y^2}{2} \right) \leq y \leq (b + 2\epsilon) \left(\frac{1 + x^2 + y^2}{2} \right). \quad (\text{B.6})$$

Now from (B.4), (B.5) we get

$$\begin{aligned} \frac{x}{\sqrt{x^2 + y^2}} - a &\leq \frac{(a + 2\epsilon)\sqrt{1 + 2\epsilon}}{(1 - 2\epsilon)^{3/2}} - a \\ &= \frac{a\sqrt{1 + 2\epsilon}}{(1 - 2\epsilon)^{3/2}} - a + \frac{2\epsilon\sqrt{1 + 2\epsilon}}{(1 - 2\epsilon)^{3/2}} \\ &\leq \left(\frac{\sqrt{1 + 2\epsilon}}{(1 - 2\epsilon)^{3/2}} - 1 \right) + \frac{2\epsilon\sqrt{1 + 2\epsilon}}{(1 - 2\epsilon)^{3/2}} \end{aligned} \quad (\text{B.7})$$

where we used $|a| \leq 1$. Note that $\sqrt{1 + 2\epsilon} \leq 1 + \epsilon$. Moreover, we have the following

Claim 1. For $0 \leq \epsilon \leq 1/4$, we have $(1 - 2\epsilon)^{-3/2} \leq (1 + 12\sqrt{2}\epsilon)$.

Proof. Consider the function $g(\epsilon) = (1 + 12\sqrt{2}\epsilon) - (1 - 2\epsilon)^{-3/2}$. We have $g'(\epsilon) = 12\sqrt{2} - 3(1 - 2\epsilon)^{-5/2}$ and so $g'(\epsilon) \geq 0$ when $0 \leq \epsilon \leq 1/4$. Thus g is increasing for this range of ϵ , and so $g(\epsilon) \geq g(0) = 0$, which completes the proof. \square

Applying these observations to (B.7), we get

$$\begin{aligned} \frac{x}{\sqrt{x^2 + y^2}} - a &\leq [(1 + \epsilon)(1 + 12\sqrt{2}\epsilon) - 1] + 2\epsilon(1 + \epsilon)2^{3/2} \\ &= [(12\sqrt{2} + 1)\epsilon + 12\sqrt{2}\epsilon^2] + 4\sqrt{2}(\epsilon + \epsilon^2) \\ &\leq [(12\sqrt{2} + 1)\epsilon + 3\sqrt{2}\epsilon] + 5\sqrt{2}\epsilon \quad (\text{Using } \epsilon \leq 1/4) \\ &\leq (20\sqrt{2} + 1)\epsilon. \end{aligned} \quad (\text{B.8})$$

The reader is invited to verify that $a - \frac{x}{\sqrt{x^2 + y^2}} \leq (20\sqrt{2} + 1)\epsilon$, which together with (B.8) implies that $|\frac{x}{\sqrt{x^2 + y^2}} - a| \leq (20\sqrt{2} + 1)\epsilon$. In an identical fashion, one obtains the same bound on $|\frac{y}{\sqrt{x^2 + y^2}} - b|$. From these observations, we then obtain

$$\chi(\alpha_1, \alpha_2/|\alpha_2|) = \frac{1}{2} \sqrt{\left(\frac{x}{\sqrt{x^2 + y^2}} - a \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}} - b \right)^2} \leq \left(20 + \frac{1}{\sqrt{2}} \right) \epsilon.$$

Using (B.2), this in turn implies that $d_w(\alpha_1, \alpha_2/|\alpha_2|) \leq (10 + \frac{1}{2\sqrt{2}})\epsilon$. \square

B.3 More useful results

Proposition 2. *For any $t_1, t_2 \in [0, 1)$, and integer n , we have*

$$|\exp(\iota 2\pi n t_1) - \exp(\iota 2\pi n t_2)| \leq 2|n| \pi d_w(t_1, t_2).$$

Proof. On one hand,

$$\begin{aligned} |\exp(\iota 2\pi n t_1) - \exp(\iota 2\pi n t_2)| &= |1 - \exp(\iota 2\pi n(t_2 - t_1))| \\ &= 2|\sin(\pi n(t_2 - t_1))| \\ &\leq 2\pi|n||t_2 - t_1| \quad (\text{Since } |\sin x| \leq |x|, \forall x \in \mathbb{R}). \end{aligned} \quad (\text{B.9})$$

On the other hand, note that

$$2|\sin(\pi n(t_2 - t_1))| = 2|\sin(\pi|n||t_2 - t_1|)| = 2|\sin(\pi|n|(1 - |t_2 - t_1|))| \leq 2\pi|n|(1 - |t_2 - t_1|). \quad (\text{B.10})$$

The bound follows from (B.9), (B.10) and by noting the definition of d_w . \square

Proposition 3. *For any $t_1, t_2 \in [0, 1)$, integer n , and $u_1, u_2 \in \mathbb{C}$, we have*

$$|u_1 \exp(\iota 2\pi n t_1) - u_2 \exp(\iota 2\pi n t_2)| \leq 2\pi|u_1||n|d_w(t_1, t_2) + |u_1 - u_2|.$$

Proof. Indeed,

$$\begin{aligned} &|u_1 \exp(\iota 2\pi n t_1) - u_2 \exp(\iota 2\pi n t_2)| \\ &= |u_1 \exp(\iota 2\pi n t_1) - u_1 \exp(\iota 2\pi n t_2) + u_1 \exp(\iota 2\pi n t_2) + u_2 \exp(\iota 2\pi n t_2)| \\ &\leq |u_1| |\exp(\iota 2\pi n t_1) - \exp(\iota 2\pi n t_2)| + |u_1 - u_2| \\ &\leq 2\pi|u_1||n|d_w(t_1, t_2) + |u_1 - u_2| \end{aligned}$$

where in the last inequality, we used Proposition 2. \square

C Conditions on ε in Theorem 5

We treat conditions 1b, 1c separately below. In this section, for a given $i \in \mathbb{N}$, we will denote $C_i(u_i, v_i, w_i, \dots)$ to be a positive term depending only on the parameters u_i, v_i, w_i, \dots .

1. Condition 1b is equivalent to

$$\frac{K}{\Delta_{L-1}} \varepsilon_{L-1} + \frac{1}{(\mu_L^2 - \mu_{L-1}^2)^{1/2}} \log^{1/2} \left(\frac{K^{3/2} \mu_L L}{\varepsilon_{L-1} \mu_{L-1}} \right) \varepsilon_{L-1} \lesssim \frac{\varepsilon_L \mu_L e^{-2\pi^2(\mu_L^2 - \mu_1^2)/\Delta_L^2}}{K^{3/2} L \mu_{L-1}} \quad (\text{C.1})$$

$$\Leftrightarrow C_1(K, \Delta_{L-1}, \mu_L, \mu_{L-1}) \log^{1/2} \left(\frac{K^{3/2} \mu_L L}{\varepsilon_{L-1} \mu_{L-1}} \right) \varepsilon_{L-1} \lesssim C_2(\mu_L, \mu_{L-1}, \mu_1, K, L, \Delta_L) \varepsilon_L. \quad (\text{C.2})$$

Since for any $0 < \tilde{\theta} < 1$, we have⁶

$$\log \left(\frac{K^{3/2} \mu_L L}{\varepsilon_{L-1} \mu_{L-1}} \right) \leq \frac{1}{\tilde{\theta}} \left(\frac{K^{3/2} \mu_L L}{\varepsilon_{L-1} \mu_{L-1}} \right)^{\tilde{\theta}}, \quad (\text{C.3})$$

⁶Here, we use the fact $\log x \leq nx^{1/n}$ for $n, x > 0$.

therefore (C.2) is satisfied if for any given $\theta \in (0, 1/2)$, it holds that

$$\varepsilon_{L-1}^{1-\theta} \lesssim \sqrt{\theta} \frac{C_2(\mu_L, \mu_{L-1}, \mu_1, K, L, \Delta_L)}{C_1(K, \Delta_{L-1}, \mu_L, \mu_{L-1})} \left(\frac{\mu_{L-1}}{\mu_L L K^{3/2}} \right)^\theta \varepsilon_L. \quad (\text{C.4})$$

From this, (3.26) follows easily.

2. We now look at condition 1c starting with the condition $E_l(\varepsilon_l) \leq E_{l+1}(\varepsilon_{l+1})$. Using the order dependency of $E_l(\cdot)$ from (3.24), this is the same as

$$C_3(K, \Delta_l, \mu_l, \mu_{l+1}) \log^{1/2} \left(\frac{K^{3/2} \mu_L L}{\varepsilon_l \mu_l} \right) \varepsilon_l \lesssim C_4(K, \Delta_{l+1}, \mu_{l+1}, \mu_{l+2}) \log^{1/2} \left(\frac{K^{3/2} \mu_L L}{\varepsilon_{l+1} \mu_{l+1}} \right) \varepsilon_{l+1}. \quad (\text{C.5})$$

Using (C.3), it follows that $E_l(\varepsilon_l) \leq E_{l+1}(\varepsilon_{l+1})$ is ensured if

$$\varepsilon_l^{1-\theta} \lesssim \sqrt{\theta} \frac{C_4(K, \Delta_{l+1}, \mu_{l+1}, \mu_{l+2})}{C_3(K, \Delta_l, \mu_l, \mu_{l+1})} \left(\frac{\mu_l}{\mu_L L K^{3/2}} \right)^\theta \log^{1/2} \left(\frac{K^{3/2} \mu_L L}{\varepsilon_{l+1} \mu_{l+1}} \right) \varepsilon_{l+1} \quad (\text{C.6})$$

holds for any given $\theta \in (0, 1/2)$.

Now consider the other condition, namely (3.23). Using the order dependency of $F_{l+1}(\varepsilon_l)$ and $E_l(\varepsilon_l)$ from (3.25), (3.24) respectively, this can be equivalently written as

$$\begin{aligned} & \left(\frac{1}{\Delta_{l+1}} + \frac{1}{(\mu_{l+2}^2 - \mu_{l+1}^2)^{1/2}} \log^{1/2} \left(\frac{K^{3/2} L \mu_L}{\varepsilon_l \mu_{l+1}} \right) \right) \varepsilon_l \\ & + \left(\frac{K}{\Delta_l} + \frac{1}{(\mu_{l+1}^2 - \mu_l^2)^{1/2}} \log^{1/2} \left(\frac{K^{3/2} L \mu_L}{\varepsilon_l \mu_l} \right) \right) \varepsilon_l \lesssim \varepsilon_{l+1} e^{-2\pi^2(\mu_{l+1}^2 - \mu_l^2) F_{l+1}^2(\varepsilon_{l+1})} \frac{\mu_{l+1}}{\mu_l K^{3/2}}. \end{aligned} \quad (\text{C.7})$$

Since $\mu_{l+1} > \mu_l$, therefore the L.H.S of (C.7) is \lesssim than

$$C_5(\Delta_l, \Delta_{l+1}, \mu_l, \mu_{l+1}, \mu_{l+2}, K) \log^{1/2} \left(\frac{K^{3/2} L \mu_L}{\varepsilon_l \mu_l} \right) \varepsilon_l. \quad (\text{C.8})$$

Moreover, since

$$2\pi^2(\mu_{l+1}^2 - \mu_l^2) F_{l+1}^2(\varepsilon_{l+1}) \lesssim C_6(\mu_1, \mu_{l+1}, \mu_{l+2}, \Delta_{l+1}) \log \left(\frac{K^{3/2} L \mu_L}{\varepsilon_{l+1} \mu_{l+1}} \right),$$

therefore the R.H.S of (C.7) is

$$\begin{aligned} & \gtrsim \varepsilon_{l+1} \left(\frac{\mu_{l+1}}{\mu_l K^{3/2}} \right) \left(\frac{\varepsilon_{l+1} \mu_{l+1}}{K^{3/2} L \mu_L} \right)^{C_6(\mu_1, \mu_{l+1}, \mu_{l+2}, \Delta_{l+1})} \\ & \gtrsim (\varepsilon_{l+1})^{1+C_6(\mu_1, \mu_{l+1}, \mu_{l+2}, \Delta_{l+1})} C_7(\mu_l, \mu_{l+1}, \mu_L, K, L). \end{aligned} \quad (\text{C.9})$$

Therefore from (C.8), (C.9), it follows that a sufficient condition for (C.7) to hold is

$$\log^{1/2} \left(\frac{K^{3/2} L \mu_L}{\varepsilon_l \mu_l} \right) \varepsilon_l \lesssim (\varepsilon_{l+1})^{1+C_6(\mu_1, \mu_{l+1}, \mu_{l+2}, \Delta_{l+1})} \frac{C_7(\mu_l, \mu_{l+1}, \mu_L, K, L)}{C_5(\Delta_l, \Delta_{l+1}, \mu_l, \mu_{l+1}, \mu_{l+2}, K)}. \quad (\text{C.10})$$

Using (C.3), we have that (C.10) holds if for any given $\theta \in (0, 1/2)$,

$$\varepsilon_l^{1-\theta} \lesssim \sqrt{\theta} \frac{C_7(\mu_l, \mu_{l+1}, \mu_L, K, L)}{C_5(\Delta_l, \Delta_{l+1}, \mu_l, \mu_{l+1}, \mu_{l+2}, K)} (\varepsilon_{l+1})^{1+C_6(\mu_1, \mu_{l+1}, \mu_{l+2}, \Delta_{l+1})} \left(\frac{\mu_l}{\mu_L L K^{3/2}} \right)^\theta. \quad (\text{C.11})$$

Comparing (C.11) and (C.6), we see that the dependence on ε_{l+1} in (C.11) is in general stricter than that in (C.6).

D Proof of Theorem 6

As for the proof of Theorem 5, the proof of Theorem 6 is divided in to three main steps. We will only point out the relevant modifications in the proof of Theorem 5 that we need to make in order to account for the noise term.

- **Recovering source parameters for first group.** For $i = -m_1, \dots, m_1 - 1$, equation (3.29) becomes

$$\frac{\tilde{f}(s_1 + i)}{\bar{g}_1(s_1 + i)} = \sum_{j=1}^K u'_{1,j} \exp(\iota 2\pi i t_{1,j}) + \eta'_{1,i} + \frac{w_1(i)}{\bar{g}_1(s_1 + i)}.$$

Since the noise term satisfies (4.2), invoking Corollary 1 brings that for $\varepsilon_1 < c\Delta_1/2$, and $\frac{2}{\Delta_1 - 2\varepsilon_1} + 1 \leq m_1 < \frac{2}{\Delta_1(1-c)} + 1 (= M_{1,\text{up}})$, we only need to impose that s_1 satisfies

$$K u_{\max}(L-1) \frac{\mu_L}{\mu_1} \exp(-2\pi^2(s_1 - m_1)^2(\mu_2^2 - \mu_1^2)) \leq \varepsilon_1 \frac{u_{\min}}{10C\sqrt{K}} \left(1 + 48 \frac{u_{\max}}{u_{\min}}\right)^{-1}$$

to recover the same error bound (3.31) as in the noiseless case. The lower and upper bounds on s_1 are then obtained in a similar way as (3.32) and (3.33) in the proof of Theorem 5, and one finally gets the same error bound (3.34) as in the noiseless case.

- **Recovering source parameters for l^{th} ($1 < l < L$) group.** Say we are at the l^{th} iteration for $1 < l < L$, having estimated the source parameters up to the $(l-1)^{\text{th}}$ group. Say that for each $p = 1, \dots, l-1$ and $j = 1, \dots, K$ the following holds.

$$d_w(\hat{t}_{p,\phi_p(j)}, t_{p,j}) \leq \varepsilon_p, \quad |\hat{u}_{p,\phi_p(j)} - u_{p,j}| < E_p(\varepsilon_p), \quad (\text{D.1})$$

for some permutations $\phi_p : [K] \rightarrow [K]$, with

1. $\varepsilon_1 \leq \dots \leq \varepsilon_{l-1}$; $E_1(\varepsilon_1) \leq \dots \leq E_{l-1}(\varepsilon_{l-1})$;
2. $\varepsilon_p < c\Delta_p/2$; $\varepsilon_q < D_q$, $2 \leq q \leq l-1$;
3. $2\pi u_{\max} F'_{q+1}(\varepsilon_q) \varepsilon_q + E_q(\varepsilon_q) \leq \varepsilon_{q+1} e^{-2\pi^2(\mu_{q+1}^2 - \mu_1^2) F_{q+1}'^2(\varepsilon_{q+1})} \frac{u_{\min} \mu_{q+1}}{15CK^{3/2} q \mu_q} \left(1 + \frac{48u_{\max}}{u_{\min}}\right)^{-1}$, $1 \leq q \leq l-2$.

For $i = -m_l, \dots, m_l - 1$, we have

$$\frac{\tilde{f}(s_l + i) - \sum_{p=1}^{l-1} \hat{f}_p(s_l + i)}{\bar{g}_l(s_l + i)} = \sum_{j=1}^K u'_{l,j} \exp(\iota 2\pi i t_{l,j}) + \eta'_{l,i,\text{past}} + \eta'_{l,i,\text{fut}} + \frac{w_l(i)}{\bar{g}_l(s_l + i)}.$$

As in the proof of Theorem 5, $\eta'_{l,i,past}$ denotes perturbation due to the estimation errors of the source parameters in the past, and $\eta'_{l,i,fut}$ denotes perturbation due to the tails of the kernels that are yet to be processed. Imposing that $s_l \geq S_l$ where

$$S_l = m_l + \frac{1}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}} \log^{1/2} \left(\frac{15CK^{3/2}u_{\max}(L-l)\mu_L}{\varepsilon_l u_{\min}\mu_l} \left(1 + \frac{48u_{\max}}{u_{\min}} \right) \right)$$

we get that, for $i = -m_l, \dots, m_l - 1$,

$$|\eta'_{l,i,fut}| < \varepsilon_l \frac{u_{\min}}{15C\sqrt{K}} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1} \quad (\text{D.2})$$

where we used the same steps as in the proof of Theorem 5.

Now, since $s_l \leq \tilde{c}S_l$ and $m_l < M_{l,\text{up}}$, we obtain

$$s_l < \tilde{c} \left(M_{l,\text{up}} + \frac{1}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}} \log^{1/2} \left(\frac{15CK^{3/2}(L-l)u_{\max}\mu_L}{u_{\min}\mu_l\varepsilon_l} \left(1 + 48\frac{u_{\max}}{u_{\min}} \right) \right) \right). \quad (\text{D.3})$$

This then gives us the bound

$$\begin{aligned} s_l + m_l &< (\tilde{c} + 1)M_{l,\text{up}} + \frac{\tilde{c}}{(2\pi^2(\mu_{l+1}^2 - \mu_l^2))^{1/2}} \\ &\quad \log^{1/2} \left(\frac{15CK^{3/2}(L-l)u_{\max}\mu_L}{u_{\min}\mu_l\varepsilon_l} \left(1 + 48\frac{u_{\max}}{u_{\min}} \right) \right) \\ &= C'_{l,1} + C'_{l,2} \log^{1/2} \left(\frac{3D_l}{\varepsilon_l} \right) = F'_l(\varepsilon_l), \end{aligned}$$

where we recall the definition of F'_l , and constants $C'_{l,1}, C'_{l,2}, D_l > 0$ from (4.1), (3.22). If $\varepsilon_l < D_l$ and $\varepsilon_l \geq \varepsilon_{l-1}$ hold, then $(s_l + m_l) < F'_l(\varepsilon_l) \leq F'_l(\varepsilon_{l-1})$. Using this in (3.37), we obtain

$$|\eta'_{l,i,past}| < \left(K(l-1) \frac{\mu_{l-1}}{\mu_l} e^{2\pi^2(\mu_l^2 - \mu_{l-1}^2)F_l'^2(\varepsilon_l)} \right) (2\pi u_{\max} F'_l(\varepsilon_{l-1})\varepsilon_{l-1} + E_{l-1}(\varepsilon_{l-1}))$$

Therefore if ε_{l-1} satisfies the condition

$$\begin{aligned} &2\pi u_{\max} F'_l(\varepsilon_{l-1})\varepsilon_{l-1} + E_{l-1}(\varepsilon_{l-1}) \\ &\leq \varepsilon_l e^{-2\pi^2(\mu_l^2 - \mu_{l-1}^2)F_l'^2(\varepsilon_l)} \frac{u_{\min}\mu_l}{15CK^{3/2}(l-1)\mu_{l-1}} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1} \end{aligned}$$

then it implies $|\eta'_{l,i,past}| < \varepsilon_l \frac{u_{\min}}{15C\sqrt{K}} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1}$. Together with (D.2), this gives

$$\left| \eta'_{l,i} + \frac{w_l(i)}{\bar{g}_l(s_l + \cdot)} \right| \leq |\eta'_{l,i,fut}| + |\eta'_{l,i,past}| + \left| \frac{w_l(i)}{\bar{g}_l(s_l + i)} \right| < \varepsilon_l \frac{u_{\min}}{5C\sqrt{K}} \left(1 + \frac{48u_{\max}}{u_{\min}} \right)^{-1}.$$

We obtain estimates $\hat{t}_{l,j}$, $\hat{u}_{l,j}$, $j = 1, \dots, K$ via the MMP method. Invoking Corollary 1, if $\varepsilon_l < c\Delta_l/2$, then for the stated choice of m_l , there exists a permutation $\phi_l : [K] \rightarrow [K]$ such that for each $j = 1, \dots, K$,

$$d_w(\hat{t}_{l,\phi_l(j)}, t_{l,j}) \leq \varepsilon_l; \quad |\hat{u}_{l,\phi_l(j)} - u_{l,j}| < (\tilde{C}_l + 2\pi u_{\max}s_l) \varepsilon_l < E_l(\varepsilon_l).$$

The last inequality follows readily using (D.3).

- **Recovering source parameters for last group.** Say that for each $p = 1, \dots, L-1$ and $j = 1, \dots, K$ the following holds.

$$d_w(\widehat{t}_{p,\phi_p(j)}, t_{p,j}) \leq \varepsilon_p, \quad |\widehat{u}_{p,\phi_p(j)} - u_{p,j}| < E_p(\varepsilon_p), \quad (\text{D.4})$$

for some permutations $\phi_p : [K] \rightarrow [K]$, with

1. $\varepsilon_1 \leq \dots \leq \varepsilon_{L-1}$; $E_1(\varepsilon_1) \leq \dots \leq E_{L-1}(\varepsilon_{L-1})$;
2. $\varepsilon_p < c\Delta_p/2$; $\varepsilon_q < D_q$, $2 \leq q \leq L-1$;
3. $2\pi u_{\max} F'_{q+1}(\varepsilon_q) \varepsilon_q + E_q(\varepsilon_q) \leq \varepsilon_{q+1} e^{-2\pi^2(\mu_{q+1}^2 - \mu_1^2) F_{q+1}'^2(\varepsilon_{q+1})} \frac{u_{\min} \mu_{q+1}}{15CK^{3/2} q \mu_q} \left(1 + \frac{48u_{\max}}{u_{\min}}\right)^{-1}$,
 $1 \leq q \leq L-2$.

We proceed by noting that for each $i = -m_L, \dots, m_L - 1$

$$\frac{\widetilde{f}(s_L + i) - \sum_{p=1}^{L-1} \widehat{f}_p(s_L + i)}{\widehat{g}_L(s_L + i)} = \sum_{j=1}^K u'_{L,j} \exp(\iota 2\pi i t_{L,j}) + \eta'_{L,i} + \frac{w_L(i)}{\widehat{g}_L(s_L + i)}.$$

Using Proposition 3, we have for each $p = 1, \dots, L-1$ and $j = 1, \dots, K$ that

$$|u_{p,j} \exp(\iota 2\pi(s_L + i)t_{p,j}) - \widehat{u}_{p,\phi_p(j)} \exp(\iota 2\pi(s_L + i)\widehat{t}_{p,\phi_p(j)})| < 2\pi u_{\max} |s_L + i| \varepsilon_p + E_p(\varepsilon_p), \quad (\text{D.5})$$

where we used (D.4). Since $s_L = 0$, hence $(s_L + i)^2 < m_L^2 < M_{L,\text{up}}^2$ for all $i = -m_L, \dots, m_L - 1$. Using (D.5), we then easily obtain as in the proof of Theorem 5

$$|\eta'_{L,i}| \leq \left(K(L-1) \frac{\mu_{L-1}}{\mu_L} e^{2\pi^2(\mu_L^2 - \mu_1^2) M_{L,\text{up}}^2} \right) (2\pi u_{\max} M_{L,\text{up}} \varepsilon_{L-1} + E_{L-1}(\varepsilon_{L-1})) \quad (\text{D.6})$$

where in the last inequality, we used (1).

We obtain estimates $\widehat{t}_{L,j}$, $\widehat{u}_{L,j}$, $j = 1, \dots, K$ via the MMP method. Invoking Corollary 1 and assuming $\varepsilon_L < c\Delta_L/2$, it follows for the stated conditions on m_L , that it suffices if ε_{L-1} satisfies

$$\begin{aligned} & 2\pi u_{\max} M_{L,\text{up}} \varepsilon_{L-1} + E_{L-1}(\varepsilon_{L-1}) \\ & \leq \varepsilon_L \frac{e^{-2\pi^2(\mu_L^2 - \mu_1^2) M_{L,\text{up}}^2} u_{\min} \mu_L}{10CK^{3/2}(L-1)\mu_{L-1}} \left(1 + \frac{48u_{\max}}{u_{\min}}\right)^{-1}. \end{aligned}$$

Indeed, combining this last bound with (4.4), there then exists a permutation $\phi_L : [K] \rightarrow [K]$ such that for each $j = 1, \dots, K$,

$$d_w(\widehat{t}_{L,\phi_L(j)}, t_{L,j}) \leq \varepsilon_L; \quad |\widehat{u}_{L,\phi_L(j)} - u_{L,j}| < \widetilde{C}_L \varepsilon_L = E_L(\varepsilon_L).$$

This completes the proof.

E Experiments

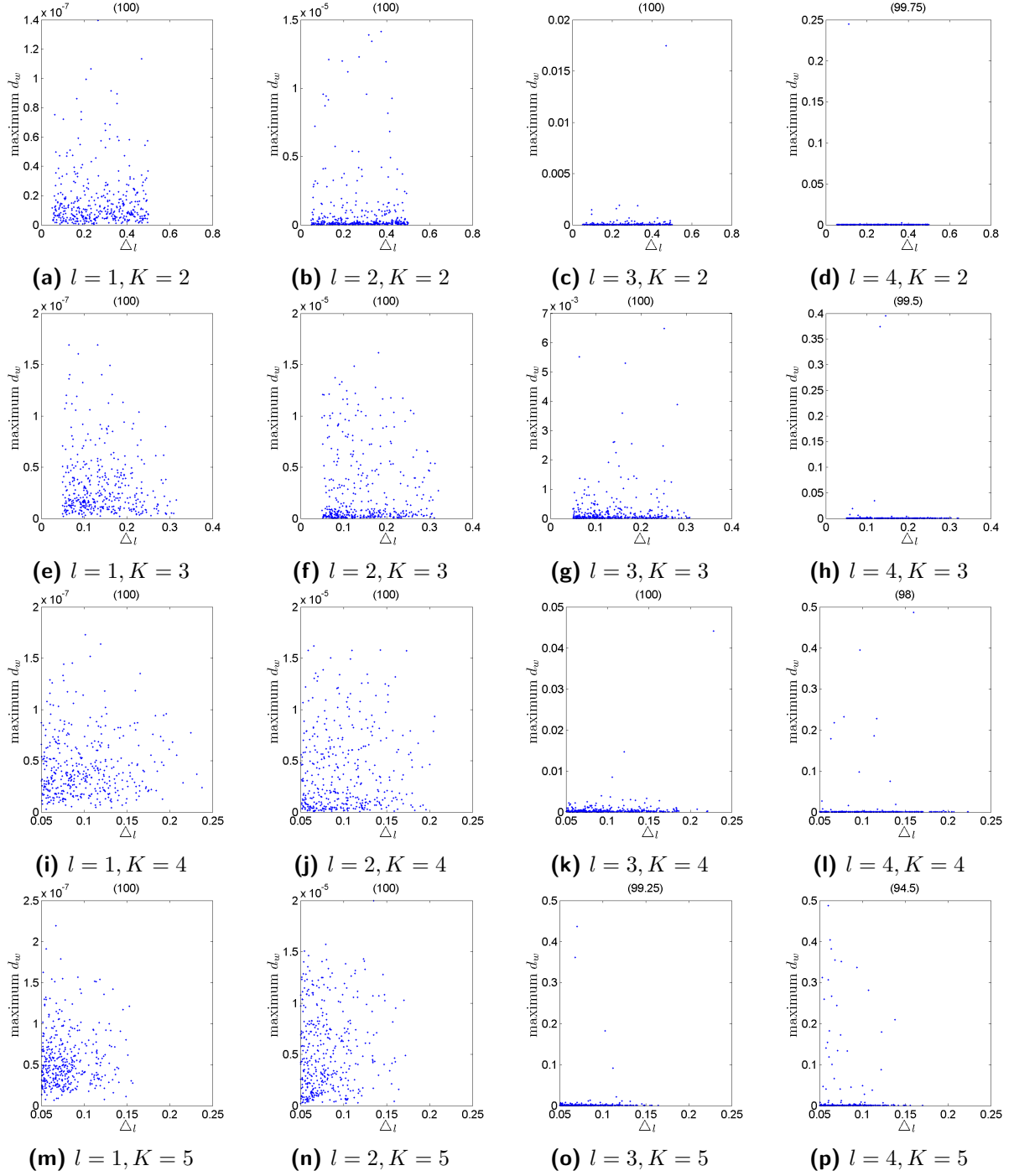


Figure 7: Scatter plots for maximum wrap around error ($d_{w,l,\max}$) v/s minimum separation (Δ_l) for 400 Monte Carlo trials, with no external noise. This is shown for $K \in \{2, 3, 4, 5\}$ with $L = 4$ and $C = 1$. For each sub-plot, we mention the percentage of trials with $d_{w,l,\max} \leq 0.05$ in parenthesis.

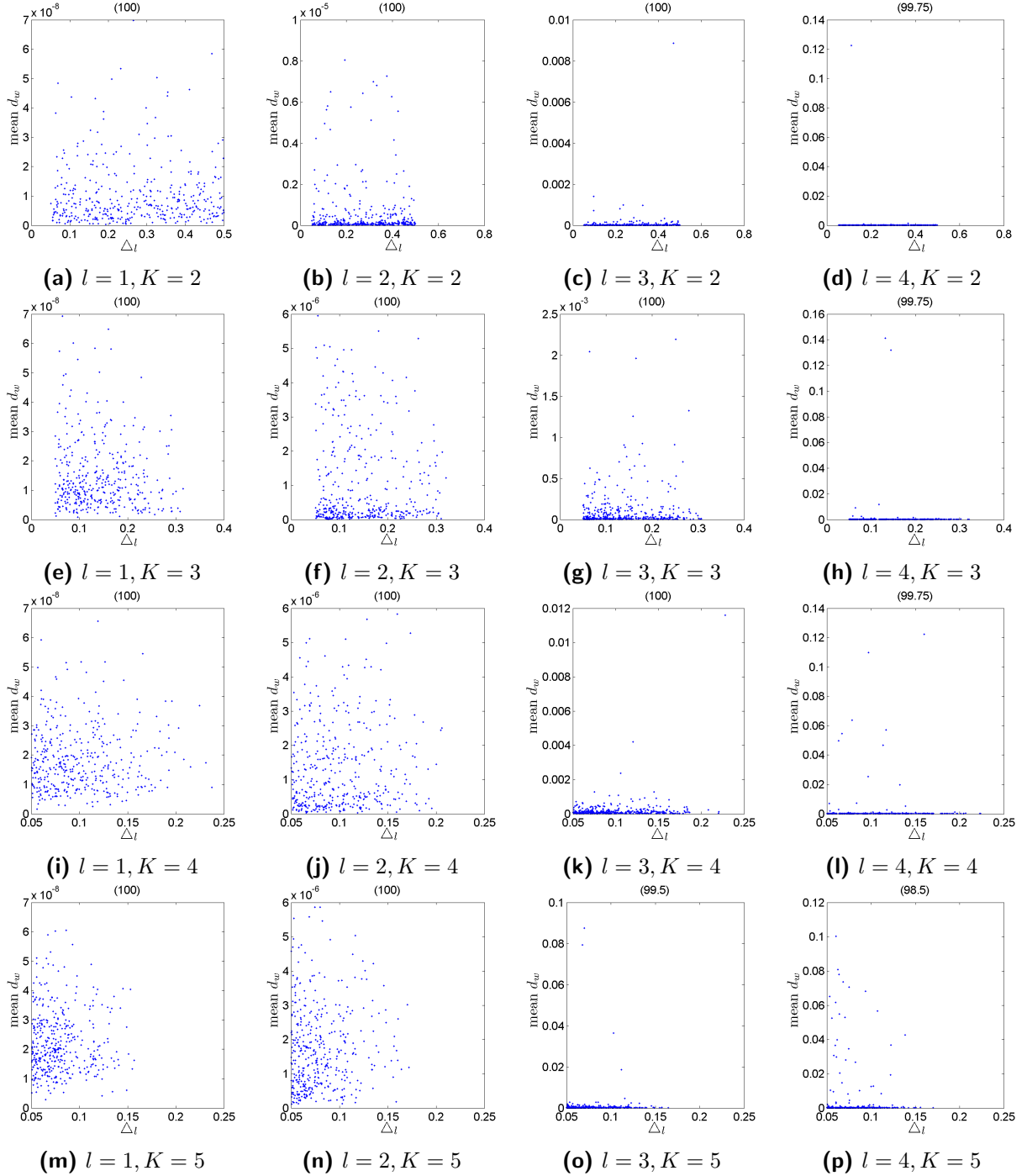


Figure 8: Scatter plots for the mean wrap around error ($d_{w,l,\text{avg}}$) v/s minimum separation (Δ_l) for 400 Monte Carlo trials, with no external noise. This is shown for $K \in \{2, 3, 4, 5\}$ with $L = 4$ and $C = 1$. For each sub-plot, we mention the percentage of trials with $d_{w,l,\text{avg}} \leq 0.05$ in parenthesis.

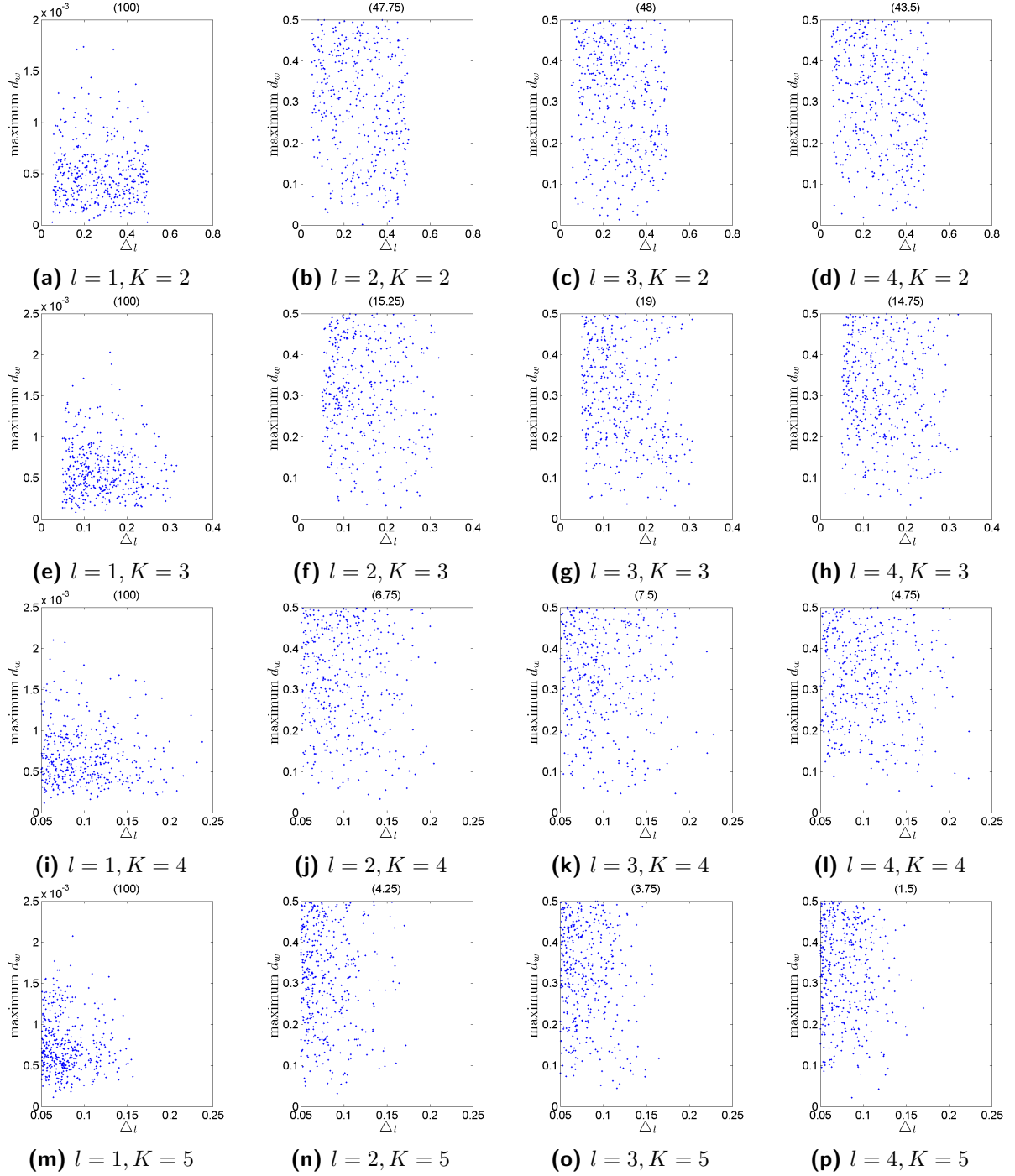


Figure 9: Scatter plots for maximum wrap around error ($d_{w,l,\max}$) v/s minimum separation (Δ_l) for 400 Monte Carlo trials, with external Gaussian noise (standard deviation 5×10^{-5}). This is shown for $K \in \{2, 3, 4, 5\}$ with $L = 4$ and $C = 1$. For each sub-plot, we mention the percentage of trials with $d_{w,l,\max} \leq 0.05$ in parenthesis.

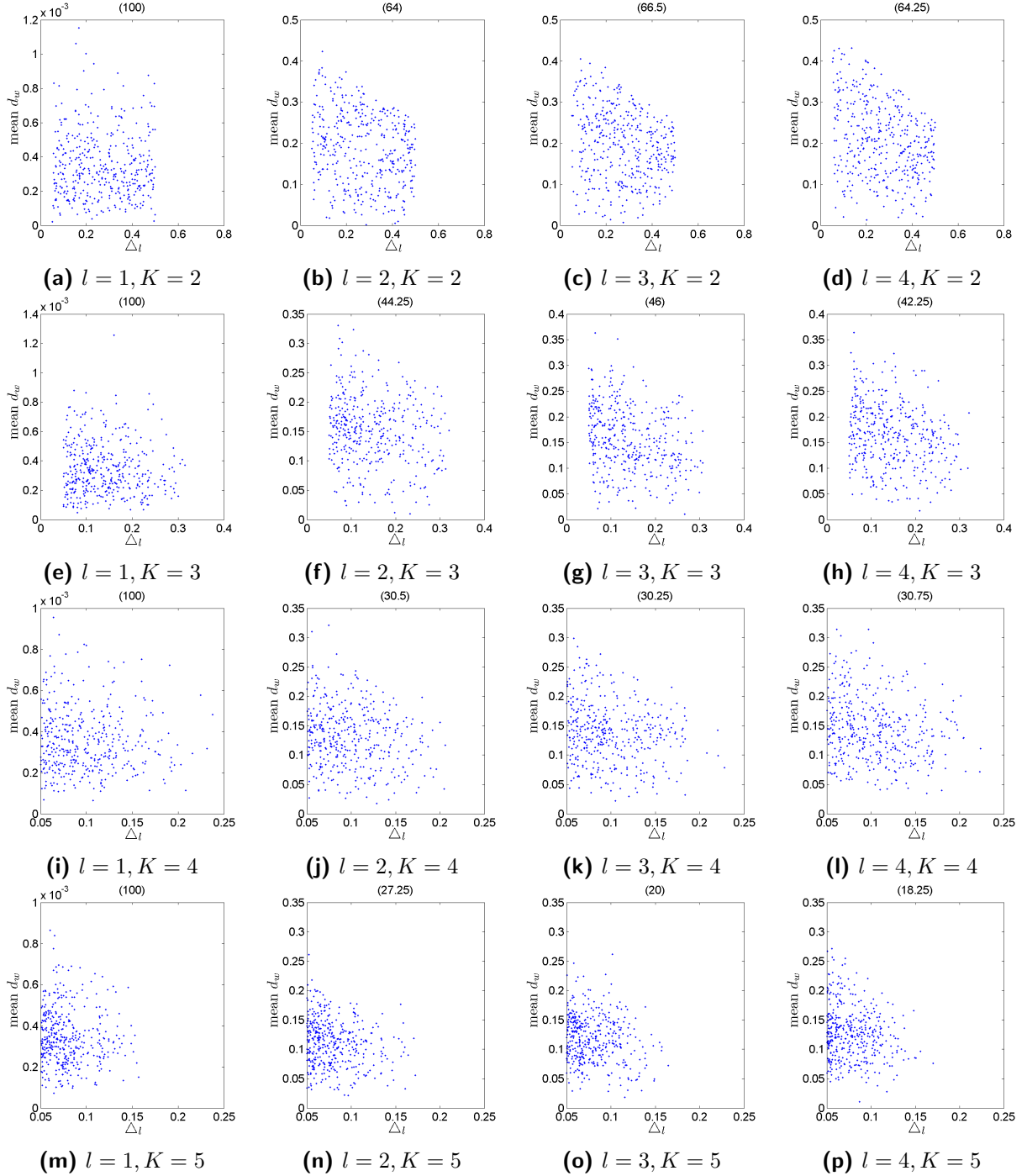


Figure 10: Scatter plots for mean wrap around error ($d_{w,l,\text{avg}}$) v/s minimum separation (Δ_l) for 400 Monte Carlo trials, with external Gaussian noise (standard deviation 5×10^{-5}). This is shown for $K \in \{2, 3, 4, 5\}$ with $L = 4$ and $C = 1$. For each sub-plot, we mention the percentage of trials with $d_{w,l,\text{avg}} \leq 0.05$ in parenthesis.